Minimum residual based model order reduction approach for unsteady nonlinear aerodynamic problems

M. Ripepi\textsuperscript{1}, R. Zimmermann\textsuperscript{2} and S. Görtz\textsuperscript{1}

\textsuperscript{1}German Aerospace Center (DLR), Institute of Aerodynamics and Flow Technology, Braunschweig, Germany
\textsuperscript{2}Institute Computational Mathematics, TU Braunschweig, Braunschweig, Germany

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Overview

1. Motivation and objectives
2. Model order reduction for unsteady CFD
3. Accelerated greedy MPE procedure
4. Aeronautical application: the LANN wing
5. Summary and outlook
Motivation

- From design to certification of an aircraft many aerodynamic data are needed – for the entire flight envelope –
- All required aerodynamic data can be derived from the pressure and shear stress distribution on the aircraft surface, from steady and unsteady simulations

Prediction of:
- Aerodynamic performance
- "Handling Qualities"
- Structural loads

Input for:
- Aerodynamic design
- Structural design and sizing
- Design of control surfaces and flight control system
- Flight simulators
Objectives

Development of a simulation environment, where ALL the necessary aerodynamic data can be determined numerically

- with high accuracy
- with Hi-Fi methods

To reduce the computational complexity, Reduced Order Models based on Hi-Fi CFD should replace (e.g. in the MultiDisciplinary Optimization) high quality models and methods
Objectives

Development of a simulation environment, where **ALL** the necessary aerodynamic data can be determined numerically

- with high accuracy
- with Hi-Fi methods

ROMs for the prediction of **steady** aerodynamic flows

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**Wide range of validity**

<table>
<thead>
<tr>
<th>Angle (°)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Full-order model, high-fidelity CFD data (system of ( n ) ODEs)</td>
</tr>
<tr>
<td>2</td>
<td>Snapshots ( S_i(p_{i1}, \ldots) )</td>
</tr>
<tr>
<td>6</td>
<td>Reduced Order Model</td>
</tr>
</tbody>
</table>

**Restricted range of validity**

- Low-dim. \((r \ll n)\) ODEs
- Description of large-scale system dynamics

**Online**

\( \alpha = 4° \)

**Prediction** \( \hat{V}(p_1, \ldots) \)

**Model order reduction approach for **unsteady** aerodynamic applications**
Model order reduction for unsteady CFD

Semi-discrete Navier-Stokes Eqs.

\[
\hat{R} \equiv R(w(t)) + \Omega \frac{\partial w(t)}{\partial t} = 0 \in \mathbb{R}^N \quad \Omega: \text{cell volumes}
\]

Search for an approximated solution \( w = [\rho, \rho v, \rho E^t] \in \mathbb{R}^N \)
- in the POD subspace \( U \in \mathbb{R}^{N \times r}, r \ll N \)
- minimizing the unsteady residual in the L₂ norm

\[
w \approx \sum_{i=1}^{r} a_i U_i + \bar{w} = Ua + \bar{w}
\]

\[
\min_a \| \hat{R}(Ua + \bar{w}) \|_{L_2}^2
\]

\( a \): unknown coefficients
\( \bar{w} = \frac{\sum_{i=1}^{m} w(t_i)}{m} \): average of the snapshots

Proper Orthogonal Decomposition

\[
Y = [w(t_1), \ldots, w(t_m)] - \bar{w}
\]

SVD: \( Y = USV^T \)

EVD: \( R = YY^T = US^2U^T \)
Model order reduction for unsteady CFD

Per each physical time step, find the POD coefficients minimizing the unsteady residual at the time step \( n+1 \), by solving a nonlinear least squares problem:

\[
\mathbf{w}_{\text{rom}}(t_{n+1}) \approx \mathbf{U} \mathbf{a} + \mathbf{w}
\]

\[
\hat{\mathbf{R}} \equiv \mathbf{R}(\mathbf{w}_{\text{rom}}(t_{n+1})) + \Omega \frac{3\mathbf{w}_{\text{rom}}(t_{n+1}) - 4\mathbf{w}_{\text{rom}}(t_{n}) + \mathbf{w}_{\text{rom}}(t_{n-1})}{2\Delta t}
\]

\[
(\mathbf{J}^T \mathbf{J} + \lambda \mathbf{I}) \Delta \mathbf{a} = -\mathbf{J}^T \hat{\mathbf{R}}
\]

Broyden’s method for updating \( \mathbf{J} \)

\[
\mathbf{J}_{k+1} = \mathbf{J}_k + \frac{\Delta \hat{\mathbf{R}} - \mathbf{J}_k \Delta \mathbf{a}}{\|\Delta \mathbf{a}\|^2} \Delta \mathbf{a}^T
\]

Updating pseudo-Hessian matrix \( \mathbf{A} \equiv \mathbf{J}^T \mathbf{J} \in \mathbb{R}^{r \times r} \Rightarrow \mathcal{O}(Nr^2) \)

\[
\mathbf{A}_{k+1} \equiv \mathbf{J}_{k+1}^T \mathbf{J}_{k+1}
\]

\[
= \mathbf{A}_k + \frac{\mathbf{J}_k^T \Delta \hat{\mathbf{R}} - \mathbf{A}_k \Delta \mathbf{a}}{\|\Delta \mathbf{a}\|^2} \Delta \mathbf{a}^T + \Delta \mathbf{a} \frac{\Delta \hat{\mathbf{R}}^T \mathbf{J}_k - \Delta \mathbf{a}^T \mathbf{A}_k \Delta \mathbf{a}}{\|\Delta \mathbf{a}\|^2} + \Delta \mathbf{a} \frac{\Delta \hat{\mathbf{R}}^T \Delta \hat{\mathbf{R}} - \Delta \hat{\mathbf{R}}^T \mathbf{J}_k \Delta \mathbf{a} - \Delta \mathbf{a}^T \mathbf{J}_k^T \Delta \hat{\mathbf{R}} + \Delta \mathbf{a}^T \mathbf{A}_k \Delta \mathbf{a}}{\|\Delta \mathbf{a}\|^4} \Delta \mathbf{a}^T
\]

\[\Rightarrow \mathcal{O}(Nr)\]

convergence

\( \mathbf{a} \leftarrow \mathbf{a} + \Delta \mathbf{a} \)
2D Test Case: NACA airfoil

- RANS equations with Spalart-Allmaras turbulence model
- Mach = 0.8, Re = 7.5 \times 10^6
- Training maneuvers exciting up to \( k \equiv \frac{\omega C}{V_\infty} = 0.2 \)
- Prediction for periodic motions at different frequencies and amplitudes (using the same ROM)

Training maneuvers

- Schroeder multisine with \( 0.01 < k < 0.2 \)
- Linear chirp signal with \( k_{\text{max}} = 0.2 \)
2D Test Case: NACA airfoil
Periodic pitching motion prediction at different oscillation frequencies

ROM built with 50 POD modes
2D Test Case: NACA airfoil
Periodic pitching motion prediction at different oscillation amplitudes

ROM built with 50 POD modes

Online Performances

<table>
<thead>
<tr>
<th></th>
<th>Total</th>
<th>Per time step</th>
<th>Speed-up</th>
</tr>
</thead>
<tbody>
<tr>
<td>TAU, $\alpha_A = 2.51^\circ$</td>
<td>15 min 58 s</td>
<td>8 min 58 s</td>
<td>1</td>
</tr>
<tr>
<td>ROM Schroeder, $\alpha_A = 2.51^\circ$</td>
<td>8 min 43 s</td>
<td>3.56 s</td>
<td>1.79</td>
</tr>
<tr>
<td>ROM chirp, $\alpha_A = 2.51^\circ$</td>
<td>8 min 43 s</td>
<td>3.49 s</td>
<td>1.83</td>
</tr>
</tbody>
</table>

Ways to improve the performances:
- **Hyper-reduction** (e.g. Gappy POD)
- Different selection of the subspace (e.g. Dynamic Mode Decomposition)
Model order reduction for unsteady CFD
Curse of dimensionality in ROMs of nonlinear systems

I. Compute the approx. solution
\[ \mathbf{w}(t) \approx \mathbf{U} \mathbf{a}(t) + \overline{\mathbf{w}} \Rightarrow O(Nr) \]

II. Evaluate the unsteady residual
\[ \hat{\mathbf{R}}(\mathbf{w}(t)) \triangleq \mathbf{R}(\mathbf{w}(t)) + \Omega \frac{\partial \mathbf{w}(t)}{\partial t} \]

III. Solve the LS problem
\[ (\mathbf{J}^T \mathbf{J} + \lambda \mathbf{I}) \Delta \mathbf{a} = -\mathbf{J}^T \hat{\mathbf{R}} \]

\[ \text{nonlinear least squares problem} \]

\[ \min_{\mathbf{a}} || \hat{\mathbf{R}}(\mathbf{Ua} + \overline{\mathbf{w}}) ||^2_{L_2} \]

Jacobian Matrix \( \mathbf{J} \in \mathbb{R}^{N \times r} \)

- \( N \): order of CFD model (variables x nodes)
- \( r \): order of the ROM (i.e. number of POD modes)

The computational cost scales linearly with the dimension \( N \) of the full order model. No significant speedup can be expected when solving the minimum residual ROM.
**Hyper-reduction approaches**

Complexity reduction by **sampling** (or compute only a few entries of) the nonlinear unsteady residual vector $\hat{R}$

- **Collocation**
  - omission of many components
  - non intrusive

- **Reconstruction**
  - approximation of the entire vector, by interpolation or by least-squares projection onto a subspace $V = U(U^TPP^TU)^{-1}U^TP$

Selecting the subset indices $\rightarrow$
- Gappy POD
- **Missing Point Estimation**
- (Discrete) **Empirical Interpolation Method**

- The complete nonlinear unsteady residual vector $\hat{R}$ is **evaluated**,
- but only a small subset of its entries are used in the **minimization** process

**Greedy:** minimize

$$\|U(U^TPP^TU)^{-1}U^TPP^T\| = 1/\sigma_{\text{min}}(P^TU)$$
Exhaustive greedy missing point estimation procedure

Greedy point selection algorithms minimize an error indicator by sequentially looping over all entries \( \rightarrow \) costly \( \Rightarrow \) > \( \mathcal{O}(Nr^3) \)

Exhaustive greedy MPE \( \rightarrow \) maximize \( \sigma_{\min}(P_{s+1}^T U) \)

**Input:**
- \( U \in \mathbb{R}^{N \times r} \): basis of a r-dim subspace
- \( J_s \in \mathbb{R}^{s \times 1} \): index set
- \( P_s \in \mathbb{R}^{N \times s} \): mask matrix with s indices, where \( s \geq r \)

**Output:**
- next index \( J_{s+1} = J_s \cup J_{opt} \)
- \( P_{s+1} T = P_s, e_j, opt \)

**Observation 1)***

\[
U^T P_{s+1} P_{s+1}^T U = \Phi_s \left( \Sigma_s^2 + v_j v_j^T \right) \Phi_s^T
\]

\[
v_j := \left( e_j^T, s+1 U \Phi_s \right)^T
\]

**Symmetric rank-one eigenvalue modification**

\[
\lambda_i = \sigma_i^2 + \mu_i \|v\|^2
\]

\[
0 < \mu_i < 1
\]

\[
\sum_{i=1}^r \mu_i = 1
\]

**The penultimate singular value bounds the growth of \( \sigma_{\min} \)**

Accelerated greedy missing point estimation procedure

**Idea:**
- Select the vectors \( \mathbf{v}_{j,\text{opt}} \) that feature the **largest absolute values in the ultimate component** while all other components are comparably small.
- Build fast approximations that sort the set of candidate vectors that induce the rank-one modifications (without solving the modified eigenvalue pb.)

**Error bound**
\[
\lambda_{r-k} < F(\sigma_{r-k}^2, \sigma_{r-k+1}^2, \mathbf{v}_i)
\]

**Fast greedy MPE** with target switching based on the **growth potential** \( \dot{O}(N_1^2) \)

**Input:** \( \mathbf{U} \in \mathbb{R}^{N \times r} \): basis of a \( r \)-dim subspace, \( \mathbf{J}_s \in \mathbb{R}^{s \times 1} \): index set, \( s \geq r \).

1. \( \bar{\mathbf{J}}_s = \{1, \ldots, N\} \setminus \mathbf{J}_s \)
2. **while** \( |\mathbf{J}_s| \leq \text{maxpoints} \) **do**
3. target the largest growth for \( \lambda_{\text{min}} \) (i.e. \( \lambda_r \)) or the next biggest \( \lambda_{r-1} \), etc.
4. determine \( j_{\text{opt}} \) by sorting \( \{\mathbf{v}_j, j \in \bar{\mathbf{J}}_s\} \) according to the fast estimate
5. update: \( \mathbf{J}_{s+1} = \mathbf{J}_s \cup \{j_{\text{opt}}\}, \quad \bar{\mathbf{J}}_{s+1} = \bar{\mathbf{J}}_s \setminus \{j_{\text{opt}}\}, \quad s = s + 1 \)

**Output:** index set \( \mathbf{J}_s \)

2D Test Case: NACA airfoil
Nodes selected with the fast greedy MPE

Total number of grid nodes: 21454
2D Test Case: NACA airfoil

Periodic pitching motion prediction at \( k = 0.16 \)

- **Maximum lift**
- **Minimum lift**

1000 nodes selected by the greedy MPE

< 5% of the total number of grid nodes
## 2D Test Case: NACA airfoil

### Offline Performances (1 core)
- Compute 400 snapshots by a training unsteady CFD simulation: ~47 min
- POD ROM building: 1 min 32 s
- Fast greedy MPE (selecting 1000 nodes): ~2 min

<table>
<thead>
<tr>
<th>Online Performances (1 core)</th>
<th>CFD TAU</th>
<th>ROM (built with 50 POD modes)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>w/o MPE</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100 nodes</td>
</tr>
<tr>
<td>CPU Time</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total [min]</td>
<td>~18</td>
<td>~9</td>
</tr>
<tr>
<td>per time step [s]</td>
<td>6.98</td>
<td>3.65</td>
</tr>
<tr>
<td>speed-up</td>
<td>1</td>
<td>1.91</td>
</tr>
</tbody>
</table>

### Process

<table>
<thead>
<tr>
<th>CPU Time</th>
<th>CFD TAU</th>
<th>ROM (built with 50 POD modes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Building the model</td>
<td>—</td>
<td>~49 min</td>
</tr>
<tr>
<td>Simulate 7 parameters (5 reduced frequencies and 2 amplitudes)</td>
<td>2 h 5 min</td>
<td>1 h 3 min</td>
</tr>
</tbody>
</table>
3D test case: LANN wing

- RANS equations with SA-neg turbulence model
- $V_\infty = 271.66$ m/s, Mach = 0.82, Re = $7.17 \times 10^6$
- Chirp training maneuver exciting up to $k \equiv \frac{\omega c_r}{V_\infty} = 0.35$
- Predicted periodic pitching oscillation: $\alpha(\tau) = 2.6^\circ + 0.25 \sin(k \tau)$, $\tau \equiv \frac{V_\infty t}{c_r}$

CFD settings
- Min residual: $1e-5$
- Max inner iterations: 100

ROM settings
- 18 POD modes
- 5000 MPE nodes
3D test case: LANN wing

Total number of grid nodes: 469213

5000 nodes selected by the greedy MPE

~1% of the total number of grid nodes
3D test case: LANN wing

**Online Performances**

- **CFD (10 cores)**
  - Total: 25 h 50 min
  - Per time step: ~16 min
  - Speed-up: 1

- **ROM (10 cores)**
  - **w/o MPE**
    - Total: 2 h 44 min
    - Per time step: 1 min 36 s
    - Speed-up: 9.5
  - **with MPE**
    - Total: 1 h 29 min
    - Per time step: 36 s
    - Speed-up: 26.7

**Offline Performances (10 cores)**

- Compute snapshots: 4d 10h 40min
- POD ROM building: ~5 min
- Fast greedy MPE: ~6 min
Summary and Outlook

The effectiveness of reduced order models for unsteady aerodynamic problems have been demonstrated for a NACA airfoil and the LANN wing in transonic flows.

Remarks

- Training maneuvers may be an issue: the quality of the resulting model will reflect the input signal quality
- Further analyses are required (e.g. moving window, full-aircraft config., ...)

Possible improvements

- Different selection of the subspace (e.g. Dynamic Mode Decomposition, Isomap)
- Different ordering of the modes, focusing more on the input-output behavior (e.g. subspace rotation)
- Hybrid strategy by switching in time ROM/CFD
- Sparsity promoting techniques (L1 norm, clustering)
Scenarios

- Multidisciplinary applications
- Real-time applications
- Design and optimization
- Probabilistic applications
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Bibliography


