Learning Development

University of Northampton

FACTORISATION

The aim of factorisation is to find 2 expressions, which, **when multiplied together**, give you the original expression.

Example

$x^2 + 2x - 3 = (x + 3)(x - 1)$

In this pack you will be shown how to do this. You will also need to remember your work on

DIRECTED NUMBERS,
RULES FOR MULTIPLYING BRACKETS
AND THE BASICS OF ALGEBRA.

Factorisation can be divided into FOUR types of questions

1. Taking out common factors.
2. Factorising by grouping.
3. Factorisation of QUADRATIC EXPRESSIONS.
4. Difference between two squares.

You are advised to make an effort to learn the examples given here!

Often in the examination, you are simply given an expression to factorise, and we think that you can easily pick up marks on this type of question.

The rules are straight-forward, so start on the next page. If there is anything which is not clear, please do not hesitate to ask for some help.

1. TAKING OUT COMMON FACTORS

In this type of question, there will be a common term in the expressions given.

Example 1

Factorise the following expression.

$2x + 2y$

1. Ask yourself if there is a number (or letter) which appears in both $2x$ and $2y$.
   Which number appears in both? **2 appears in both**.

2. Now put this number outside the bracket, like this:-
   2 (   )

3. In effect you are dividing the original number by the common factor, which we have said is 2.

   If you divide $2x$ by 2, you are left with x.
   If you divided $2y$ by 2, you are left with y.

   The x and the y are then placed INSIDE THE BRACKET, like this:-

   $2(x + y)$ which is your answer.

   To check that you have the correct answer, (and that the signs are correct!) multiply what is inside the bracket by what is in front of the bracket:

   $2(x + y)= 2x + 2y$ answer, therefore, is correct

Example 2

Factorise the following:-

3a + 6

Ask yourself , "Is there a common factor?" 3 appears in both numbers.
(3 can be divided by 3, and 6 can be divided by 3.)

Write the common factor in front of the brackets, 3( ) divide the original numbers by the common factor

$3a \div 3 = a$ and $6 \div 3 = 2$

So, **a** and **2** must go inside the bracket, like this:-

3(a + 2) which is the answer

To check, multiply the terms inside the bracket by what is in front of the bracket. In doing this, you should arrive back at the original expression you were given.

3(a + 2) = 3a + 6 **CORRECT!**

Example 3

Factorise: $mn^3 – mn$

Remember that this means (m x n x n x n) - (m x n)

Ask yourself, "Which term(s) appear(s) in both?" m and n appear in both, and must, therefore, be placed in front of the bracket.

mn(   )

Divide the first term by mn, which gives n x n (or n^2) **DIVIDE THE SECOND TERM BY** mn, which gives -1.

Now write these inside the bracket.

mn (n^2 - 1) which is the answer.

Now check by multiplying the terms inside the bracket by what is in front of the bracket.

$mn(n^2 -1) = mn^3 - mn$

Please make a note at this stage, that 1 (the number one) is a very important number. When every other idea that you have tried fails, then try 1!

Example 4

$5x + 10x^2 - 15x^3$
$= 5x(1 + 2x - 3x^2)$

Please try these.

Look back over the pack to refresh your memory!

Exercise 1

Factorise the following:

1. 5a + 5b 2. 3a -12

3. $b^2 + 4b$ 4. ab - ay

5. $2b - 4b^2$ 6. $4x^2 + 16x$

7. $a^3 + a^2 + a$ 8. $2a^3 - 4a^2 + a$

9. $4b^3 - 8b^2 + 12b$ 10. $36c^3 - 9c$

If you are not sure at this stage, PLEASE ASK FOR SOME HELP!

2. MULTIPLICATION OF TWO BRACKETS

When there are two brackets to be multiplied together, what we want is all the terms in the first bracket multiplied by all the terms in the second bracket. (Do you agree? If not, ask for an explanation.)

Example 5

$(x + 2)(3x - 5)$

We want to multiply $(x + 2)$ by $(3x - 5)$ **OR** $(3x - 5)$ by $(x + 2)$.

Whichever way it is done, makes no difference to the result.

First Step

Write out the second bracket TWICE**,** like this:

$(3x - 5)\ \ (3x - 5)$

leaving space in between the brackets.

Second Step

Write the first term in the first bracket IN FRONT of the first $(3x - 5)$

$x(3x - 5)\quad (3x - 5)$

Then write out the second term in the first bracket in front of the second $(3x - 5)$, to give

$x(3x - 5) + 2(3x - 5)$

Third Step

Multiply out the two brackets. This gives
$3x^2 - 5x + 6x - 10$

Now gather together the like terms, to give:

$3x^2 + x - 10$ (which is the answer)

(You will be shown later in this pack how to check the answer.)

Example 6

$(x + y)^2$ which means $(x + y)(x + y)$

First Step

Write out the second bracket twice,

$(x + y) \quad (x + y)$

Second Step

The x from the first (original) bracket is written in front of the first $(x + y)$, and the y from the original first bracket is written in front of the second $(x + y)$, like this:-

$x(x + y) + y(x + y)$

Third Step

Multiply out the bracket, to give:

$x^2 + xy + y\,x + y^2$

Remember that xy is the same as $y\,x$, so in this example we have $2xy$

$x^2 + 2xy + y^2$ is the answer

Example 7

$(x - y)^2$

Can also be written $(x - y)(x - y)$

Instead of you being given each step, see if you can see what is happening in the example below. Try to relate it to the three steps you were given in the previous two examples.

Step 1 $\quad (x - y) \quad (x - y)$
Step 2 $\quad x(x - y) \quad - y(x - y)$
Step 3 $\quad x^2 - xy \quad - y\,x + y^2$

Answer = $x^2 - 2xy + y^2$

Now try these - remove the brackets in the way you were shown above.

Exercise 2

1. (a + 2) (a + 4) 2. (a + 2) (a - 4)

3. (a - 2)(a + 4) 4. (a - 2)(a- 4)

5. (a + 3)(2a - 6) 6. $(a + b)^2$

7. $(a - b)^2$ 8. $(2x + 3y)^2$

9. $(x - y)(x + y)$ - SPECIAL EXAMPLE, which you will see again later in this pack. Try to remember it!

10. $(2x - 3y)(x + 3y)$

STOP HERE IF YOU ARE ENTERING AT LEVEL 1

3. FACTORISATION BY GROUPING

This type of factorisation is easily identified, because there are always 4 TERMS.

Example 8

Factorise this expression

ax + ay +bx + by

First Step

Divide the four terms into TWO PAIRS, like this:-

+ax + ay + bx + by

Second Step

Find common factor(s) for each pair of numbers.

+a(x + y)   +b(x + y)

Third Step

The brackets should read the same - in this case (x + y).
Write this in ONE BRACKET on the next line, and write another pair of brackets:

(x+ y)(   )

INSIDE the empty bracket, write the terms which are outside the bracket at the second step - in this case +a and +b

(x + y)(a + b) is the answer

It does not matter about the order in which you write the brackets. If you were to multiply the two brackets you would obtain the same result.

Example 9

a + b + ay – by

First Step

Divide into two pairs

a + b   - ay - by

Second Step

Look for common factors - in this case, 1 is common to the 1st pair of numbers AND -y is common to the 2nd pair which gives

+1(a + b)   -y(a +b)

Look carefully at the signs, and where 1 is used.

Third Step

The terms INSIDE BOTH brackets should be the same.
Write this inside one pair of brackets on the next line, then write what is OUTSIDE the BRACKETS in another bracket.

Answer = (a + b)(1 - y)

Starting from the beginning, this question would be set out like this:-

a + b - ay - by
= a + b   - ay - by
= 1(a + b) - y(a +b)
= (a + b)(1 -y)

Example 10

Sometimes the grouping has already been done for you, as in this example below:

a(2x - 1) – b(2x - 1)
Answer =   (2x - 1)(a -b)

Example 11

Sometimes the terms need to be re-arranged, before they can be grouped:

$px + qy + qx + py$

Go through the steps given below

First Step

Divide the fours terms into 2 pairs

$px + qy \quad + qx + py$

Second Step

Take out common factors from each pair. You can see that there are no common factors, when the four terms are arranged like this. You must RE-ARRANGE the terms so that there is a COMMON FACTOR in each pair of terms, like this:

$px + qx + py + qy$

Now you can go back to the third step, and take out common factors, giving:

$x\,(p + q) + y(p + q)$
$= \quad (p + q)(x + y)$

NOTE

It does not matter how you arrange the terms. The important thing is that each pair of terms has a common factor. The above example COULD have been arranged as follows:-

$+px + py \quad + qx + qy$

$= p(x + y) \quad +q(x + y)$

$= (x + y)(p + q)$

Which is the same as the previous arrangement.

Exercise 3

1. $px + py + q\,x + qy$
2. $px - py - qx + qy$
3. $a(b + 1) - 3\,(b + 1)$
4. $x^2(y - 10) -6(y - 1)$
5. $2ab - 4ac + bd - 2dc$
6. $am + bn + bm + an$

3. FACTORISATION OF QUADRATIC EXPRESSIONS

This is AN IMPORTANT SECTION with which you are advised to become very familiar.

The work you do here will be connected with quadratic equations.

What is the difference between a quadratic expression and a quadratic equation?

quadratic expression $x^2 + 3x - 2$ (has no = sign)
quadratic equation $x^2 + 3x$ x $-2 = 0$ (has terms on both sides of the = sign)

Note that both contain a 'squared' term

Let us look at an example to give you an idea of what is to be done!

Example 12

Factorise: $x^2 + 7x + 12$

The answer to this must be two expressions in brackets, which when multiplied together give the original question (above).

In this case, the answer is

$(x + 4)(x + 3)$

Work carefully through the explanation, **which** can be applied to every question of this type.

Steps to follow:-

1. Look at the co-efficient of the squared term. The co-efficient of x^2 is 1 because it could have been written $1x^2$.

2. Look at the last term, which is +12.

3. Multiply the co-efficient of the squared term by the last term:

   +1 multiplied by +12 which is +12.

4. Now find the factors of +12.

+12	-12	+6	-6	+4	+4
+1	-1	+2	-2	+3	-3

(+12) x (+1) =
(-12) x (-1) =
(+6) x (+2) = All give the answer +12
(-6) x (-2) =
(+4) x (+3) =
(-4) x (-3) =

5. Now add the pairs of factors. This is to find the middle term. In this case $+7\chi$.

+12 +1 = +13
-12 - 1 = - 13
+6 + 2 = + 8
-6 - 2 = - 8
+4 + 3 = + 7
-4 - 3 = - 7

The only pair of numbers which gives + 7, is + 4 + 3.

6. Now look back at question $x^2+7x+12$

Write down the first term and the last term, with their (correct) signs, leaving a space in between them.

$X^2 \quad + \; 12$

7. Write in the space, the numbers which you found in step 5 (+4 and +3). There are four terms, so divide them into two pairs, as you did earlier in this pack (in factorising by grouping):

$x^2 + 4x \qquad + 3x + 12$

8. Find common factors of each pair.

$x(x + 4) \qquad +3(x + 4)$

$+ x$ is common +3 is common

9. Term inside two brackets are same. Write this term in one bracket. Write terms which are outside the brackets in a second bracket, as follows:

$(x + 4)(x + 3)$

10. Check your answer by multiplying out the two brackets.

$(x + 4)(x + 3) = x^2 + 7x + 12$

Example 13

Factorise: $x^2 + 9x + 20$

1. Write down co-efficient of the squared term---------------- 1
   Write down the last term-- +20

2. Multiply these two numbers (+1) x (+20) = +20

3. Find the factors of + 20

+ 20	-20	+10	-10	+5	-5
+1	-1	+2	-2	+4	-4

   You have multiplied each pair of numbers to give +20.

4. Now add the pairs of numbers to obtain the middle term of the original equation (in this case +9).

   +20 + 1 = +21
   -20 - 1 = -21
   +10 + 2 = +12
   -10 - 2 = -12
   +5 + 4 = +9
   -5 - 4 = - 9

The pair of number which when added together, gives +9 is +5 and +4.

5. Look at the original equation $x^2 + 9x + 20$
   Write down the squared and the last term, leaving a space in between.

   $X^2 \quad + \quad 20$

6. Now write in the two terms you found in step 5. There will be four terms, so divide them into two pairs of terms:

   $x^2 + 5x \qquad +4x + 20$

7. Look for common terms in each pair

   $x^2 + 5x \qquad +4\chi + 20$
   $+x$ is common   +4 is common

8. Write down the common factors outside the brackets, to give:

   $x(x + 5) \qquad +4(x + 5)$

9. The terms inside the brackets are the same.

   $(x + 5)(x + 4)$ = **correct answer**

10. Check: $(x + 5)(x + 4) = x^{2+}9x + 20$

Example 14

Factorise $x^{2}-2x-15$

Follow the steps given above.

Suggested lay-out!

We need factors of -15

Rough work (-15)

-15	+15	-5	+5
+1	-1	+3	-3

when added give
-14 +14 -2 +2

Therefore, (-5) x (+3) =-15

and -5 + 3 = -2 (middle term)

x^2 -15

$= x^2 - 5x$ $+3x - 15$
$+ x$ is common +3 is common

$= x(x - 5)$ $+3(x - 5)$ SIGNS!

Answer $= (x - 5)(x + 3)$

Example 15

Factorise $10x^2 + 19x - 15$

(+10) (-15)
Factors of -150

Rough work

-150	+150	-75	+75	-50	+50	-25	+25
+1	-1	+2	-2	+3	-3	+6	-6

Which term when added gives the middle term +19?

+25 and -6 = +19

Write in first and last term.

$10x^2 - 15$

Write in +25 and -6. Divide the four terms into 2 pairs!

$10x^2 + 25x$ $-6x$ -15
$5x$ is common -3 is common

Write common terms in front of bracket:

$5x(2x + 5)$ $-3(2x + 5)$
Answer $= (2x + 5)(5x - 3)$

Example 16

$12x + 11x - 15$

$(+12) \times (-15)$
Find factors of -180

Rough work

+180 -180 +90 -90 +60 -80 +45 -45 +38 -36 +30 -30 +20 - 20 +18 -18

-1 +1 -2 +2 -3 +3 -4 +4 -5 +5 -6 +6 -9 +9 -10 +10

Which pair, when added, gives the middle term, +11? **+20 and -9**

$12x^2 \qquad - 15$

Write in $+20x$ and $-9x$

$12x^2 + 20x \qquad -9x - 15$
$4x$ is common $\qquad$ -3 is common

Write the common terms outside the bracket.

$4x(3x + 5) \qquad -3(3x + 5)$ SIGNS!

Answer $= (3x + 5)(4x - 3)$

Exercise 4

Factorise:

1.	$x^2 + 5x + 6$	2.	$x^2 + 7x +.12$
3.	$x^2 - x - 12$	4.	$x^2 - 2x - 15$
5.	$x^2 + 8x + 15$	6.	$x^2 - 8x + 15$
7.	$x^2 + 2x + 1$	8.	$x^2 - 4x - 21$
9.	$x^2 + 4x - 21$	10.	$x^2 - 12x + 20$
11.	$x^2 - 4x - 5$	12.	$x^2 + 8x + 7$
13.	$x^2 - 11x + 30$	14.	$x^2 - 11x + 28$
15.	$x^2 - 4\chi + 4$	16.	$3x^2 + 5x + 2$
17.	$3x^2 - 5x + 2$	18.	$2x^2 + 7x - 4$
19.	$2x^2 + 7x + 6$	20.	$6x^2 - 13x + 2$

4. DIFFERENCE BETWEEN TWO SQUARES

You can easily recognise this typo of factorisation because there are ONLY TWO terms.

You will be dealing with the **square roots** of numbers and letters.

Remember

$\sqrt{100} = \sqrt{10}^{\,2} = 10$ and $\sqrt{\chi}^{\,2} = \chi$

Example 17

Factorise $(x^2 - y^2)$

Which becomes $(x - y)(x + y)$ = Answer

$\sqrt{\chi}^{\,2} = x$ and the $\sqrt{y}^{\,2} = y$

Write the square roots of the letters as shown below:

$(x \quad y)(x \quad y)$

A minus sign is placed in one bracket and a plus sign is placed in the other, giving the answer

$(x - y)(x + y)$

Check – multiply out these brackets to give

$x^2 + xy - xy - y^2$

$x^2 - y^2$ (No middle term, and original expression.)

Example 18

Factorise: $(x^2 - 9)$

First, find the $\sqrt{\chi}^{\,2}$, which is x then $\sqrt{9}$, which is 3.

Now write two brackets, as follows:

$(\quad)(\quad)$

Write in x as the first term in each bracket, and then, write 3 as the last term in each bracket.

$(x \quad 3)(x \quad 3)$

In the first bracket write a minus sign, and in the second bracket, write a plus sign.

$(x - 3)(x + 3)$

Check by multiplying out the brackets, as you have been shown. The result is (x^2 - 9) – NO middle term.

Example 19

Factorise: 9 x^2 - 25

First find the square roots of all the numbers and letter in the question.

$\sqrt{9}$ = 3

$\sqrt{\chi}\,^2 = x$

$\sqrt{25} = 5$

Write two brackets
(   )(   )

Write in the square roots of each number and letter (3x 5)(3x 5)

Write in a negative sign in one bracket and a positive sign in the other bracket
(3x + 5)(3x - 5)

NB - It does not matter if you put the signs the other way round. The result is the same - it gives no middle term, and the squares of the numbers.

Example 20

Factorise: 18c² - 50

You must first take out the common factor of 2, to give:

2(9c² - 25)

Now find the square roots of all the terms, to give the answer:

2(3c - 5)(3c + 5)

Please ask if you need an explanation!

Example 21

Factorise: 3x^2 - 108

Take out common factors. In this case, 3 is the only common factor. This gives

3(x^2 - 36)

Now find $\sqrt{\chi}\,^2$ which is x and $\sqrt{36}$ which is 6.

Answer = 3(x + 6)(x - 6)

The difference between two squares can be used to simplify arithmetical calculations.

Example 22

Pythagoras' Theorem

Hypotenuse^2 - side^2

$$78^2 - 22^2 = (78 + 22)(78 - 22)$$
$$= 100 \times 56$$
$$= 5600$$

Beware $x^2 + y^2$ **cannot** be factorised

Remember this! It often appears in examinations.

Exercise 5

1. $(x^2 - 4)$
2. $(x^2 - 49)$
3. $(x^2 - 64)$
4. $(x^2 - 100)$
5. $(2x^2 - 200)$
6. $(4x^2 - 25)$
7. $(9x^2 - 36y^2)$
8. $(a^2 - b^2)$
9. $(25x^2 - 100y^2)$
10. $(a^2 + b^2)$

Learning Development

University of Northampton

ANSWERS

Exercise 1 TAKING OUT COMMON FACTORS

1. $5(a + b)$
2. $3(a- 4)$
3. $b(b +4)$
4. $a(b - y)$
5. $2b(1 – 2b)$
6. $4x(x + 4)$
7. $a(a^2 + a + 1)$
8. $a(2a^2 – 4a +1)$
9. $4b(b^2 – 2b + 3)$
10. $9c(4c^2 - 1)$

Exercise 2 REMOVING BRACKETS

1. $a^2 + 6a + 8$
2. $a^2 - 2a - 8$
3. $a^2 + 2a - 8$
4. $a^2 – 6a + 8$
5. $2 a^2 + 18$
6. $a^2 + 2ab + b^2$
7. $a^2 - 2ab + b^2$
8. $4x^2 + 12xy + 9y^2$
9. $x^2 – y^2$
10. $2 x^2 +3 xy – 9y^2$

Exercise 3 FACTORISATION BY GROUPING

1. $(x + y)(p + q)$
2. $(x - y)(p - q)$
3. $(b + 1)(a - 3)$
4. $(y - 1)(x^2 - 6)$
5. $(b – 2c)(2a + d)$
6. $(a + b)(m + n)$

Learning Development

University of Northampton

Exercise 4 QUADRATIC EXPRESSIONS

1.	$(x + 3)(x + 2)$	2.	$(x + 4)(x + 3)$
3.	$(x - 4)(x + 3)$	4.	$(x - 5)(x + 3)$
5.	$(x + 5)(x + 3)$	6.	$(x - 5)(x - 3)$
7.	$(x + 1)(x + 1)$ "x + 1 squared"	8.	$(x - 7)(x + 3)$
9.	$(x + 7)(x - 3)$	10.	$(x - 10)(x - 2)$
11.	$(x - 5)(x + 1)$	12.	$(x + 7)(x + 1)$
13.	$(x - 5)(x - 6)$	14.	$(x - 7)(x - 4)$
15.	$(x - 2)(x - 2)$ "x – 2 squared"	16.	$(3x + 2)(x + 1)$
17.	$(3x - 2)(x - 1)$	18.	$(2x - 1)(x + 4)$
19.	$(2x + 3)(x + 2)$	20.	$(6x - 1)(x - 2)$

Exercise 5 DIFFERENCE BETWEEN TWO SQUARES

1.	$(x - 2)(x + 2)$	2.	$(x - 7)(x + 7)$
3.	$(x - 8)(x + 8)$	4.	$(x + 10)(x + 10)$
5.	$2(x - 10)(x + 10)$	6.	$(2x - 5)(2x + 5)$
7.	$9(x - 2y)(x + 2y)$	8.	$(a - b)(a + b)$
9.	$25(x - 2y)(x + 2y)$	10.	Not Possible