DISTANCE, VELOCITY AND ACCELERATION

DISTANCE-TIME GRAPHS

'Rates of change', starts with a distance s against time t graph. The gradient of the graph $\frac{ds}{dt}$ at a point gives the speed of the object at that instant.

The distance variable is in fact often called x, so x be used from now on. Now, instead of y following x, it will be x following t: that is, t will be the independent variable, and x will be worked out from t, and will therefore be the dependent variable.

The gradient of a distance-time graph will be $\frac{dx}{dt}$.

The speed of an object is commonly referred to as velocity, and is therefore given the name v. 'Velocity' is a more technical term than 'speed', and strictly speaking has a direction as well as size. As we shall be dealing only with motion in a straight line for the moment, directions such as N, S, E, W will not concern us. But is important to be conscious of the positive and negative directions of distance and speed.

The diagram shows 0 as the zero point of x. The arrow points in positive x.

If the object is moving in the direction of the arrow then x is increasing, and the velocity is positive. If the object is moving in the positive direction then x is decreasing, and the velocity is negative.

The velocity is the rate of change of distance: $v = \frac{dx}{dt}$

So if the connection between x and t is $x = 7 + 3t - t^2$

then $v = \frac{dx}{dt} = 3 - 2t$

So when $t = 0$ then $x = 7$ $v = 3$ (moving in direction of arrow)

when $t = 1$ then $x = 9$ $v = 1$ (moving in direction of arrow)

when $t = 2$ then $x = 9$ $v = -1$ (moving in opposite direction to arrow)

when $t = 5$ then $x = -3$ (has moved back past the point 0)

$v = -7$ (moving in opposite direction to arrow)

and so on.

From these figures it would be reasonable to guess that x has a maximum when $t = 1.5$; and indeed $\frac{dx}{dt}$ $(= v)$ is zero when $t = 1.5$.

The maximum value of x is 9.25.

Notice that no unit has been mentioned. In fact variables are always pure numbers; but you will usually be concerned with practical situations, so you will normally have some units in mind. In this case, if x is in metres and t is in seconds then strictly speaking you should say: $v = 30$ means that the velocity is 30 m/s which can also be written as: $v = 30$ m/s

Exercise 1

1. Here is an equation of motion:

   $x = 7 + 12t - 2t^2$

   x is in metres and t is in seconds.

   Calculate the velocity when t is:

   a) 0
   b) 1
   c) 2
   d) 3
   e) 5
   f) 10 seconds.

   In each case state whether the object is moving in the direction of the arrow (towards positive x) or in the opposite direction (towards negative x)

2. The distance x of an object from a point, in metres, is given by $x = 15t^3 - t^5$ where t is the time in seconds.

   Calculate the velocity of the object when the time t is:

   a) 0
   b) 1
   c) 2
   d) 3
   e) 4 seconds.

   What is the maximum distance from the point during this time?

Now check your answers.

ACCELERATION

The second derivative of x as a function of t is $\frac{d^2x}{dt^2}$. A name has been given to $\frac{dx}{dt}$; it is called v.

So $\frac{d^2x}{dt^2}$ is the same as $\frac{dv}{dt}$.

$\frac{dv}{dt}$ is the rate of change of velocity.

One of the figures often quoted for cars is the time to go from 0 to 60 mph. If this time is 10 seconds, then the average rate of change of velocity is 60 mph/second (6 miles per hour per second). This is, of course, the **acceleration**.

In the technical and scientific worlds the more usual unit of distance is the metre. Then the unit of acceleration is metres per second per second; it could be written as $m/s/s$ or $ms^{-1}s^{-1}$, but it is usually written as m/s^2 or ms^{-2}, and referred to as metres per second squared.

The letter a is often used as the acceleration variable; so

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

NOT DECELERATION

The word 'acceleration' has been adopted; the word 'deceleration' has **NOT**. Whatever the variations of motion of an object, its rate of change of speed is **acceleration**. You may therefore, if you wish, use the word 'deceleration' to a friend, an acquaintance, or even a stranger, when referring to the slowing down of a car; but the word should not be used in maths, science or technology.

It is not even true that deceleration is negative acceleration. The sign of a (acceleration) depends purely on which direction is defined as positive. For example, suppose x is defined as positive upwards; if you fall out of a helicopter your acceleration will be negative – not that you would use the word 'deceleration' in those circumstances!

SOME EXAMPLES

Example 1

$$x = 7 + 3t - t^2$$

$$v = \frac{dx}{dt} = \frac{d^2x}{dt^2} = -2$$

So the acceleration is constant at -2 m/s^2. The velocity falls by 2 m/s every second. When t is zero v is 3 m/s, so in successive seconds v would be 1, -1, -3, -5, -7 m/s and so on.

Example 2

$$x = 5 + 3t^2$$

Then $$v = \frac{dx}{dt} = 6t$$

And $$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 6$$

When t is zero the object has a velocity of zero but an acceleration of 6 m/s^2.

An object in free fall has a constant acceleration downwards of (about) 9.8 m/s^2. If you throw a stone straight upwards, it will slowdown, stop, and then start downwards.

During this time it is a constant acceleration of 9.8 m/s^2 downwards, even while it is stationary at the top of its path.

Exercise 2

In the following, distance is in metres and time is in seconds.

1. Distance $x = 8t - t^2$

   Calculate the distance, velocity and acceleration when

   a) $t = 0$
   b) $t = 1$
   c) $t = 2$
   d) $t = 5$
   e) $t = 10$

2. Distance $x = 108t - t^4$

   Find the time when x is a maximum, the maximum value of x, and the acceleration at that time.

3. When an object moves vertically under gravity, its height x is given by $x = ut - 0.5gt^2$ if the air resistance can be neglected.

   x is measured vertically **upwards,** u is the starting velocity and g is the acceleration due to gravity, about 9.8 m/s^2.

   a) From the equation for x, obtain formulas for the velocity and the acceleration of the object. Is the acceleration always the same (independent of t)?

   b) If u is 30 m/s, find the maximum value of x.

   c) Calculate the velocity after 2, 6 and 10 seconds. What does the negative velocity imply?

4. The distance x of an object from its starting point is given by $x = 7.3t - 0.8t^3$

   a) find the formulas for the velocity and the acceleration;

   b) find the initial (starting) velocity and acceleration;

   c) find the velocity and acceleration after 3 seconds.

Now check your answers.

SUMMARY

For motion in a straight line, distance is x and time is t.

t is the independent variable, x is the dependent variable: $x = f(t)$.

The velocity v is given by $v = \frac{dx}{dt}$.

The acceleration is given by $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.

Velocity has direction as well as magnitude: positive velocity means motion in the direction of the increasing x; negative velocity means motion in the direction of decreasing x.

Acceleration also has direction as well as magnitude. The direction depends on how positive x has been defined, not on whether the object is slowing down or speeding up.

ANSWERS

Exercise 1

1. Using the basic rule:

First, differentiate to get the formula for v, then put in the values of t.

$x = 7 + 12t - 2t^2$

$v = \frac{dx}{dt} = 12 - 4t$

Now there is a formula for v. Given any of t, v can be found.

a) $t = 0$ gives $v = 12$ Velocity is 12 m/s, object moves in direction of arrow.

b) $t = 1$ gives $v = 8$ Velocity is 8 m/s, object moves in direction of arrow.

c) $t = 2$ gives $v = 4$ Velocity is 4 m/s, object moves in direction of arrow.

d) $t = 3$ gives $v = 0$ Stationary.

e) $t = 5$ gives $v = -8$ Velocity is -8 m/s, object moves in opposite direction.

f) $t = 10$ gives $v = -28$ Velocity is -28 m/s, object moves in opposite direction.

2. The maximum distance is 162m. Remember, you are looking for a time when $v = 0$.

$x = 15t^3 - t^5$

$v = \frac{dx}{dt} = 45t^2 - 5t^4$

Here is a table:

	a)	b)	c)	d)	e)
t	0	1	2	3	4
v	0	40	100	0	-560

$\frac{dx}{dt}$ is zero when $t = 0$ and when $t = 3$

You can tell that from the table, so you do not need to solve the equation. But to solve it anyway:

The equation to solve is $45t^2 - 5t^4 = 0$

The key point is that 5 and t^2 are factors, so the equation becomes

$5t^2(9 - t^2) = 0$ which gives $t = 0$ or $t^2 = 9$, $t = \pm 3$

The -3 solution is outside the range of interest. The solutions inside the range are t = 0 and t = 3

When t = 0, then x = 0, so this is so obviously not the maximum distance.

Does t = 3 give a maximum for x?

$\frac{d^2y}{dx^2} = 90t - 20t$. When t = 3 then $\frac{d^2y}{dx^2}$ = 270-540.

So $\frac{d^2y}{dx^2}$ is indeed negative when t =3, so this gives a maximum of x.

So maximum x is $15 \times 3^3 - 3^5$ = 162.

The maximum distance is 162 m.

Now return to the text.

Exercise 2

1. The answers are given in the table below.

$x = 8t - t^2$

First find the formulas for v and a.

$v = \frac{dx}{dt} = 8 - 2t$

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = -2

Using the formulas, I can fill in the table:

t	0	1	2	5	10
x	0	7	12	15	-20
v	8	6	4	-2	-12
a	-2	-2	-2	-2	-2

2. Maximum x is 243 m.

If you have any difficulty with these, please refer back to pack ………It contained some very similar problems.

$x = 108t - t^4$

$v = \frac{dx}{dt} = 108 - 4t^3$ $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -12t^2$

$\frac{dx}{dt}$ is zero when $4t^3 - 108 = 0$

$4t^3 = 108$

$t^3 = 27$ giving $t = 3$

So x is stationary when $t = 3$. The acceleration a is always negative, so this must be a maximum (this needs to be stated as part of the answer).

Then maximum $x = 108 \times 3 - 3^4 = 243$ m.

and when $t = 3$ then $a = -12 \times 3^2 = 108\ m/s^2$.

3. By convention the letter u is used for the initial (starting) velocity of an object, so this is a case where a letter from the second half of the alphabet is used to represent a constant. There is no problem with g. So concentrate on the t terms, and keep the constants as multipliers.

$v = u = -0.5\ gt^2$

a) $v = \frac{dx}{dt} = u - gt$ $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -g$

The acceleration does not depend on t. It has a constant value of $-g$.

b) If $u = 30$ and $g = 9.8$, then $x = 30t - 4.9t^2$

and $v = \frac{dx}{dt} = 30 - 9.8t$

So $\frac{dx}{dt} = 0$ when $30 - 9.8t = 0$, $t = \frac{30}{9.8} = 3.06$

$\frac{d^2x}{dt^2}$ is always negative, so this is a maximum.

So maximum height $= 30 \times 3.06 - 4.9 \times 3.06^2$

$= 45.9$ m

c) $v = 30 - 9.8t$

So when $t = 2$, $v = 10$ m/s

when $t = 6$, $v = -28.8$ m/s

when $t = 10$, $v = -68$ m/s

Since x is measured positive upwards, a negative velocity means that the object is travelling downwards.

4. $x = 7.3t - 0.8t^3$

a) $v = \dfrac{dx}{dt} = 7.3 - 2.4t^2$

$a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2} = -4.8\,t$

b) The initial values are the values when $t = 0$

So initial $v = 7.3$

initial $a = 0$

c) When $t = 3$ then $v = 7.3 - 21.6 = -14.3$ m/s and $a = -4.8 \times 3 = -14.4$ m/s.

Now return to the text.