Learning Development

University of Northampton

GRADIENTS BY FORMULA

GRADIENT AT THE POINT (*x, y*)

Now let's see about getting a formula for the gradient $\frac{dy}{dx}$, given that the formula for *y* is $y = x^2$.

Start at the point A (x, y), where $y = x^2$.

Increase the x coordinate by a small amount h to get to the point B.

Then the new $y = (x+h)^2 = x^2 + 2hx + h^2$

So change in x coordinate is *h*.

Change in y coordinate is new y – old y = $x^2 + 2hx + h^2 - x^2$
= $2hx + h^2$

$$\text{Gradient} = \frac{\text{change in y}}{\text{change in x}} = \frac{2hx + h^2}{h} = 2x + h$$

As *h* tends to zero the gradient tends to $2x$.

So when $y = x^2$

The formula for the gradient is $\frac{dy}{dx} = 2x$.

So when $x = 2$ then $\frac{dy}{dx} = 4$ (as you've already guessed).

When $x = 3$ then $\frac{dy}{dx} = 6$

When $x = 7$ then $\frac{dy}{dx} = 14$ and so on.

Exercise 1

Gradient that $y = x^2$, copy the following table and then fill in the blanks.

x	1		2.5		8		-5
y	1			16			
$\frac{dy}{dx}$		4				20	

Now check your answers.

DIFFERENTIATION

The process of obtaining a formula for the gradient is known as **differentiation**. This is another of those words borrowed from the English language (fraction, factor, multiply, divide, etc.) to be used with a very specific meaning in mathematics: which shouldn't be confused with everyday meaning ('can't you differentiate between adulteration and adultery Mr. Robinson?').

So 'differentiate x^2 with respect to x' requires the result $2x$ (differentiating with respect to other variables comes later).

Or you might see something like this: $y = x^2$

Differentiating,

$$\frac{dy}{dx} = 2x$$

The result of differentiating is something called the **differential,** sometimes the **differential coefficient** (rather confusingly, because coefficient usually means the multiplier in front of a term). Sometimes (probably most often) the **derivative.**

Differentiating x^3

The point A is (x, y), where $y = x^3$.

To get to the point B, increase x by the small amount h. New y is $(x+h)^3$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

So change in y is new y – old y = $x^3 + 3x^2h + 3xh^2 + h^3 - x^3$

$$= 3x^2h + 3xh^2 + h^3$$

Gradient is $\dfrac{\text{change in } y}{\text{change in } x} = \dfrac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2$

As h tends to zero, gradient tends to $3x^2$

So if $y = x^3$, $\dfrac{dy}{dx} = 3x^2$

Exercise 2

a) Find the gradient of the curve y = x^3 when $x = -4$ and when $x = 5$.

b) Find the point (x, y) on the curve $y = x^3$ where x is positive and $\dfrac{dy}{dx} = 12$ (in other words, find the value of x and the value of y).

Now check your answers.

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FROM FIRST PRINCIPLES

This method of finding the gradient is often referred to as 'from first principles'. If a question says 'find the gradient from first principles', it means do it, by making a small change in x or whatever the horizontal variable is called), and then making the change tend to zero.

In this pack the small change is called '*h*'. In many books the small change is called δx (pronounced 'delta exe') and the corresponding change in y is called δy (delta wye). So δx and δy are definite small changes, while dx and dy are infinitesimal changes. Then you might see in a book:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

This means that $\frac{dy}{dx}$ is the value that $\frac{\delta y}{\delta x}$ gets closer and closer to as δx

approaches zero. Mathematicians put it like this because they don't like denominators to actually equal zero.

Using δx and δy in this pack has been avoided because the use of the Greek letter δ (delta) adds an unnecessary air of mystery and 'advanced maths' to the subject.

SUMMARY

The method of finding the gradient by making a small change in x, then letting the change go towards zero, can be used to find a formula for $\frac{dy}{dx}$.

For $y = x^2$ this method gives $\frac{dy}{dx} = 2x$

And for $y = x^3$ it gives $\frac{dy}{dx} = 3x^2$.

This method is described as finding $\frac{dy}{dx}$ 'from first principles'.

In maths books the small changes are usually called 'delta exe' and 'delta wye', written with the Greek letter delta: $\delta x, \delta y$.

When you have formulas for y and $\frac{dy}{dx}$, you can not only work out y from x, but you can also immediately work out the gradient, that is, the rate at which y is increasing (or decreasing).

OTHER POWERS OF x

What about y = x^n?

If we can extend the result to x^n, we will have formulas for a huge range of mathematical expressions.

Put $y = x^n$ (don't forget n is a constant; it's temporarily standing in for a number such as 5)

Again start at the general point A (x, y), where $y = x^n$

For the point B, increase x by the small amount h

Then new $y = (x + h)^n$

Now the fact is that when you multiply out $(x + h)^n$ it goes like this:

$$(x + h)^n = x^n + nx^{(n-1)}h + \text{terms involving higher powers of h}$$

For example, $(x + h)^2 = x^2 + 2xh + h^2$

$$(x + h)^3 = x^3 + 3x^2h + \ldots$$

$$(x + h)^4 = x^4 + 4x^3h + \ldots$$

(If you're not convinced, try a few more)

So change in y is new y – old y $= x^n + nx^{(n-1)}h + \ldots - x^n$

$$= nx^{(n-1)}h + \ldots$$

So gradient $= \dfrac{\text{change in y}}{\text{change in } x} = \dfrac{nx^{(n-1)}h}{h} + \ldots$

$$= nx^{(n-1)} + \text{terms in h}, h^2, \text{etc.}$$

Now as h tends to zero, all the terms involving h and higher powers of h disappear, and what's left is $\dfrac{dy}{dx}$

So if $y = x^n$ then $\dfrac{dy}{dx} = nx^{(n-1)}$

This is an absolutely fundamental result which you must remember!

If you want to be **different**
Use the power as the coefficient.
But the power hasn't gone
It's just been reduced by one.

('differentiate' would not work in the verse so 'different' will have to do.)

Here are some examples:

$y = x^4,\ \frac{dy}{dx} = 4x^3$ $\qquad$ $y = x^7,\ \frac{dy}{dx} = 7x^6$

$y = x^{19},\ \frac{dy}{dx} = 19x^{18}$ $\qquad$ $y = x^{157},\ \frac{dy}{dx} = 157\,x^{156}$

Exercise 3

Check your answer to a), then continue if it's correct. If not, look through the text again.

a) $y = x^5,\ \frac{dy}{dx} =$ $\qquad$ b) $y = x^9,\ \frac{dy}{dx} =$

c) $y = x^{12},\ \frac{dy}{dx} =$ $\qquad$ d) $y = x^{21},\ \frac{dy}{dx} =$

e) Find the gradient of the curve $y = x^3$ when $x = 4$.

f) Find the gradient of the curve $y = x^6$ when $x = 2$.

g) Find the gradient of the curve $y = x^5$ when $x = -3$.

Now check your answer.

The case of $y = ax^n$

Remember a and n represent constants, so $y = ax^n$ could stand for
$y = 3x^7$ or $y = 19x^2$ and so on.

If $y = ax^n$ instead of x^n, then every y value is multiplied by a. In that case **changes** in y are multiplied by a; so the gradient must also be multiplied by a.

It therefore follows that if $y = ax^n$, then $\frac{dy}{dx} = anx^{(n-1)}$

So if $y = 3x^5$, then $\frac{dy}{dx} = 3 \times 5x^4 = 15x^4$

If $y = 7x^2$, then $\frac{dy}{dx} = 7 \times 2x = 14x$

If $y = \frac{x^6}{4}$ that's the same as $y = \frac{1}{4}x^6$

So $\frac{dy}{dx} = \frac{1}{4} \times 6x^5 = \frac{6}{4}x^5 = \frac{3}{2}x^5$

(there are a number of correct ways of writing the last answer:

$\frac{3}{2}x^5$ is the same as $\frac{3x^5}{2}$ and $1.5x^5$. 1 ½ x^5 is also correct form, but it is more likely to be misread than other forms.)

You may find it helpful in this and later problems to think of the coefficient (the multiplier) as something which can be ignored while the rest of the expression is dealt with, and then brought back.

So if y = $3x^5$ think of y = $3x^5$. Concentrate on the x^5: differentiating produces $5x^4$. Bring back the 3:

$$\frac{dy}{dx} = 3\,(5x^4) = 15x^4$$

Exercise 4

Find $\frac{dy}{dx}$ in the following cases:

Check your answers to a) before continuing.

a) y = $7x^2$ b) y = $5x^4$

c) y = $-3x^8$ d) y = $0.2x^{10}$

e) y= x^4 f) y = $-1.4x^{12}$

g) y = $8x$ h) y = $\frac{3x^5}{7}$

j) Find the gradient of the curve y = when $6x^3$ when x = 5.

k) Find the gradient of the curve y = $\frac{2x^4}{5}$ when x = 0.4.

Now check your answers.

SUMMARY

The derivative of x^n is nx^{n-1}. Or in other words, if y = x^n, then $\frac{dy}{dx}$ = nx^{n-1}. For example, the derivative of x^6 is $6x^5$.

The derivative of ax^n, where a constant, is anx^{n-1}.

For example, the derivative of $4x^6$ is 4 x $6x^5$ = $24x^5$.

When you want to find the gradient of a curve at a certain point, first of all you find the derivative. Then you put the value of x (or whatever the variable is called) into the formula, to get the gradient.

POLYNOMIALS

A polynomial is a collection of powers of x (or whatever variable name you happen to be using).

The expression $-3x^2 + 6 + 5x^4 - 2x$ is a polynomial. It is more usual to write the terms in order of powers of x – either increasing or decreasing; so an equation for y would be written:

$y = 5x^4 - 3x^2 - 2x + 6$

$y = 6 - 2x - 3x^2 + 5x^4$

(not all the powers of x from the lowest to the highest level have to be present; there is no x^3 term in this polynomial)

For a change on x, the change in y is simply the total (allowing for pluses and minuses) of the changes of the individual terms. So the derivative of y is the total of the derivatives of the individual terms:

$$\frac{dy}{dx} = 20x^3 - 6x - 2$$

Exercise 5

a) Find the $\frac{dy}{dx}$ in the following cases:

i) $y = 3x^5 + 8x + 5$

ii) $y = -2x^4 + 7x^3 - 3x^2 - 6x - 12$

iii) $y = 4x + 5x^2 - 2x^3 + 3x^4 - x^5 + 2x^5$

iv) $y = \frac{3x^4}{8} + \frac{x^3}{4} - 7x^2 + \frac{2x}{3} - \frac{5}{12}$

b) If $y = 2x^4 + 7x^3 + 5x^2 + 20x + 9$, find the gradient of the curve

i) when $x = -1$

ii) when $x = 0$

iii) when $x = 2$

c) If $y = x^2 - 6x - 8$, find what value of x makes $\frac{dy}{dx} = 0$

d) Find $\frac{ds}{dt}$ if $s = 3t^4 - 7t^3 + t - 6$. If s represents distance in metres and t represents time in seconds, find the speed on m/s when t is 2 seconds.

Now check your answers.

A polynomial is a collection of positive powers of x (or any other variable) such as
$3 - 2x + 5x^2 - 7x^5$.

The derivative of a polynomial is found simply by differentiating each of its terms:

If $y = 3 - 2x + 5x^2 - 7x^5$

Then $\frac{dy}{dx} = -2 + 10x - 35x^4$

NEGATIVE POWERS

The rule for negative powers is exactly the same as for positive powers.

For example, if $y = x^{-4}$, then $\frac{dy}{dx} = nx^{n-1} = -4x^{-5}$

If $y = 7x^{-4}$, then $\frac{dy}{dx} = 7 \times (-4x^{-5}) = -28x^{-5}$

If $y = -6x^{-8}$, then $\frac{dy}{dx} = (-6) \times (-8x^{-9}) = 48x^{-9}$

So the calculus is the same, but you need to be confident about your handling of negative numbers and negative powers.

Exercise 6

Find $\frac{dy}{dx}$ in the following cases:

a) $y = 3x^{-2}$

b) $y = -5x^{-3}$

c) $y = 4x^{-1} + 7x$

c) $y = \frac{3}{x^2} - \frac{15}{x} + \frac{x^2}{4}$

Now check your answers.

FRACTIONAL POWERS

Once again, **EXACTLY THE SAME RULE** applies. And any errors are likely to come from the fractions, not from the calculus.

For example, if y = $x^{3/2}$ then $\frac{dy}{dx} = \frac{3}{2} x^{1/2}$

If y = $x^{2/3}$ then $\frac{dy}{dx} = \frac{2}{3} x^{-1/3}$ because $\frac{2}{3}$ -1 = -$\frac{1}{3}$

If y = $\frac{3}{4} x^{2/5}$ then $\frac{dy}{dx} = \frac{3}{4} \times \frac{2}{5} x^{-3/5} = \frac{3}{10} x^{-3/5}$

Exercise 7

Find $\frac{dy}{dx}$ in the following cases:

a) y = $x^{7/3}$

b) y = $x^{-1/4}$

c) y = 6 $x^{3/4}$

d) y = $\frac{2}{5} x^{5/7}$

Now check your answers.

SUMMARY

The rule for differentiating the negative and the fractional powers is exactly the same as that for positive integer powers.

Negative and fractional numbers are trickier to deal with than positive integers, so this is where you have to be careful.

ANSWERS

Exercise 1

Here is the completed table, with the answers in bold type:

x	1	**2**	2.5	4	8	**10**	-5
y	1	**4**	**6.25**	16	**64**	**100**	**25**
$\frac{dy}{dx}$	2	4	**5**	**8**	**16**	20	**-10**

The most likely error here is putting the answer in the wrong place in the table: for example, in the second column along, putting 2 in the *y row* instead of the *x row*.

It's easy to make the old mistake of saying $(-5)^2$ is -25.

If you noticed that if $y = 16$, then x can be -4 as well as 4 then you're really on the alert! If x is -4 then of course $\frac{dy}{dx} = -8$.

Now return to the text.

Exercise 2

a) First find $\frac{dy}{dx}$, then put in the value of x.

If $y = x^3$, then $\frac{dy}{dx} = 3x^2$

So when $x = -4$, $\frac{dy}{dx} = 3 \text{ x } (-4)^2 = 3 \text{ x } 16 = 48$.

And when $x = 5$, $\frac{dy}{dx} = 3 \text{ x } 5^2 = 3 \text{ x } 25 = 75$.

b) If $y = x^3$, then $\frac{dy}{dx} = 3x^2$

So $\frac{dy}{dx} = 12$ means $3x^2 = 12$

If $3x^2 = 12$ then $x^2 = 4$ and $x = \pm 2$ (it's very easy to forget the negative answer!)

However the question states that x is positive, so the value for x is 2,
and so $y = 2^3 = 8$

So the point where the gradient of the curve $y = x^3$ is 12 is the point (2, 8).

Now return to the text.

Exercise 3

a) $\frac{dy}{dx} = 5x^4$ b) $\frac{dy}{dx} = 9x^8$

c) $\frac{dy}{dx} = 12x^{11}$ d) $\frac{dy}{dx} = 21x^{20}$

If you get any of these wrong, then you should read through the section again. In the rest find the formula for $\frac{dy}{dx}$, then put the values of x into the formula. (You've seen that before and you'll see it again. A mistake that people make occasionally is to put the value of x into the formula for y, and then try to find $\frac{dy}{dx}$.)

e) $y = x^3$ $\frac{dy}{dx} = 3x^2$. When $x = 4$ $\frac{dy}{dx} = 3 \text{ x } 4^2 = 48$

If you made a slip in the arithmetic then take care. Check your answers.

If you don't understand the problems, try working through the section again. It is important to get this right before you move on.

f) $y = x^6$ $\frac{dy}{dx} = 6x^5$

when $x = 2$ $\frac{dy}{dx} = 6 \text{ x } 2^5 = 6 \text{ x } 32 = 192$

when $x = -1$ $\frac{dy}{dx} = 6 \text{ x } (-1)^5 = 6 \text{ x } (-1) = -6$

g) $y = x^5$ $\frac{dy}{dx} = 5x^4$

when $x = -3$ $\frac{dy}{dx} = 5\text{v } (-3)^4 = 5 \text{ x } 81 = 405$

when $x = 0$ $\frac{dy}{dx} = 5 \text{ x } 0^4 = 0$

when $x = 2$ $\frac{dy}{dx} = 5 \text{ x } 2^4 = 5 \text{ x } 16 = 80$

Exercise 4

a) $\frac{dy}{dx}$ = 7 x 2x = 14x

Check this one to see if you're on the right track.

b) $\frac{dy}{dx}$ = 5 x 4x^3 = 20x^3

c) $\frac{dy}{dx}$ = (-3) x 8x^7 = -24x^7

d) $\frac{dy}{dx}$ = 0.2 x 10x^9 = 2x^9

e) $\frac{dy}{dx} = \frac{1}{8}$ x $4x^3 = \frac{1}{2}x^3 = \frac{x}{2}$ (this answer can be written either way)

f) $\frac{dy}{dx}$ = (-1.4) x 12x^{11} = -16.8x^{11}

g) $\frac{dy}{dx}$ = 8

h) $\frac{dy}{dx} = \frac{3}{7}$ x $5x^4 = \frac{15x^4}{7}$

j) First find the formula for $\frac{dy}{dx}$, then put the value of x into the formula.

$\frac{dy}{dx}$ = 6 x 3x^2 = 18x^2

When x = 5 $\frac{dy}{dx}$ = 18 x 5^2 = 18 x 25 = 450

k) Same again

$\frac{dy}{dx} = \frac{2}{5}$ x $4x^3 = \frac{8}{5}x^3 = \frac{8x^3}{5}$

When x = 0.4 $\frac{dy}{dx} = \frac{8x(0.4)^3}{5}$ = 0.1024

Now return to the text.

Exercise 5

a) i) $\frac{dy}{dx} = 15x^4 + 8$

If your answer's right, then the rest of these should be straightforward. If it is wrong find out why.

ii) $\frac{dy}{dx} = -8x^3 + 21x^2 - 6x - 6$

iii) $\frac{dy}{dx} = 4 + 10x - 6x^2 + 12x^3 - 5x^4 + 12x^5$

iv) $\frac{dy}{dx} = \frac{3x^3}{2} + \frac{3x^2}{4} - 14x + \frac{2}{3}$

b) First find the formula for $\frac{dy}{dx}$, then put the values of x into the formula.

$\frac{dy}{dx} = 8x^3 - 21x^2 + 10x + 20$

i) When $x = -1$ $\frac{dy}{dx} = -8 - 21 - 10 + 20 = -19$

ii) When $x = 0$ $\frac{dy}{dx} = 20$

iii) When $x = 2$ $\frac{dy}{dx} = 64 - 84 + 20 + 20 = 20$

c) First find $\frac{dy}{dx}$

$\frac{dy}{dx} = 2x - 6$

So $\frac{dy}{dx} = 0$ when $2x - 6 = 0$ that is when $x = 3$

d) First find $\frac{ds}{dt}$

$\frac{ds}{dt} = 12t^3 - 21t^2 + 1$

The next part of the question is just asking you to put t into the equation.

When t = 2 $\frac{ds}{dt}$ = 12 x 2^3 – 21 x 2^2 + 1

= 12 x 8 – 21 x 4 + 1

= 13

So speed = 13m/s when time is 2 seconds.

Note that it is better to think of formulas as containing pure numbers (no units). In this case, having worked out that $\frac{ds}{dt}$ = 13, we can then say that the speed is 13m/s.

If you are getting these problems substantially correct, then you have grasped the essentials of calculus. If you are having trouble, it may be that you are simply expecting it to be hard. There are no hidden complications here.

Exercise 6

a) $\frac{dy}{dx}$ = 3 x (-2)x^{-3} = -6x^{-3}

b) $\frac{dy}{dx}$ = (-5) x (-3)x^{-4} = 15x^{-4}

c) $\frac{dy}{dx}$ = 4 x (-1)x^{-2} + 7 = -4x^{-2} + 7

(as the powers of x get more complicated, it's easy to forget that the derivative of 7x is just 7)

It's also easy to forget that x^{-2} means $\frac{1}{x^{2'}}$, and $\frac{1}{x^2}$ means x^{-2}

d) y = 3x^{-2} - 15x^{-1} + $\frac{1}{4}x^2$

So $\frac{dy}{dx}$ = 3 x (-2)x^{-3} - 15 x (-1)x^{-2} + $\frac{1}{4}$ x 2x

= -6x^{-3} + 15x^{-2} + $\frac{x}{2}$

Now return to the text.

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Exercise 7

a) $\frac{dy}{dx} = \frac{7}{3}x^{4/3}$

b) $\frac{dy}{dx} = -\frac{1}{4}x^{-5/4}$

c) $\frac{dy}{dx} = 6 \times \frac{3}{4}x^{-1/4} = \frac{9}{2}x^{-1/4}$

d) $\frac{dy}{dx} = \frac{2}{5} \times \frac{5}{7}x^{-2/7} = \frac{2}{7}x^{-2/7}$