Learning Development

University of Northampton

HARMONIC MOTIONS

SINE WAVES AND SIMPLE HARMONIC MOTION

Here's a nice simple fraction: y = sin (x)

Differentiate $\frac{dy}{dx}$ = cos (x)

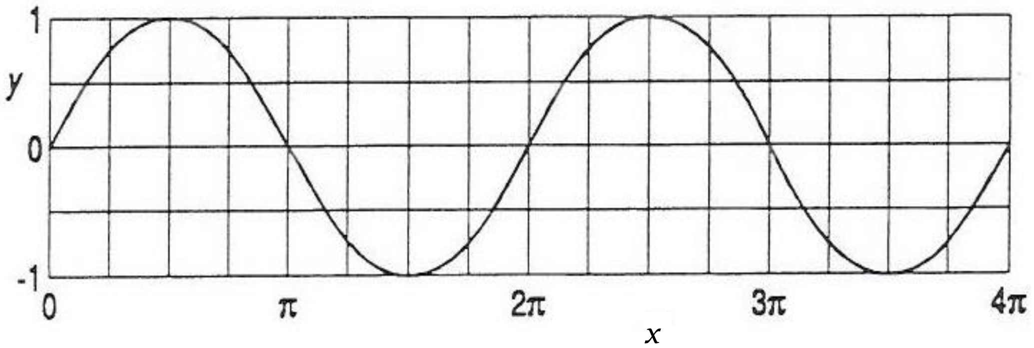
So sin (x) has a stationary value whenever cos (x) = 0.

That's when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, and so on.

Maximums or minimums?

$$\frac{d^2y}{dx^2} = -\sin(x)$$

So when sin (x) is positive then $\frac{d^2y}{dx^2}$ is negative, indicating a maximum;
and when sin (x) is negative then $\frac{d^2y}{dx^2}$ is positive, indicating a minimum. These are easily recognisable features of sine wave.



Here's another equation giving distance x in terms of the time t:

$$x = B \sin(\omega t)$$

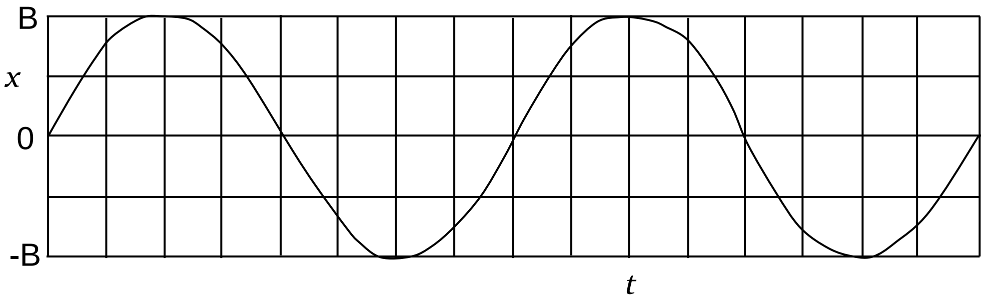
This is the well-known equation of Simple Harmonic motion (though you may be more familiar with it in connection with a.c theory in electricity). It describes the motions of weight on the end of a spring, a weight on the end of a simple pendulum, a molecule of air as a pure sound tone is transmitted, and other natural occurrences.

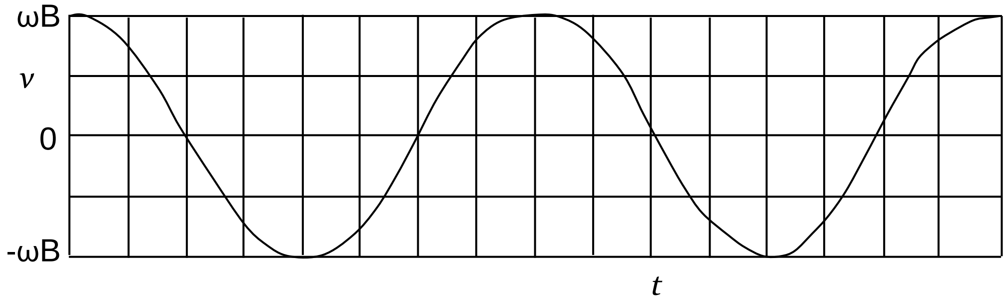
ω is the Greek letter omega, and it is traditionally used in this equation; it is sometimes called the angular frequency. That's because, if you think of a sine wave as being generated by a rotating arm, ω is the angular velocity of the arm in radians per second.
Since the angle ωt increases by 2π radians during each complete cycle of the sine wave, $\omega = 2\pi f$, where f is the frequency.
Then constant B is the amplitude if the motion, and is the furthest distance reached from the origin (the origin being the point where x = 0)

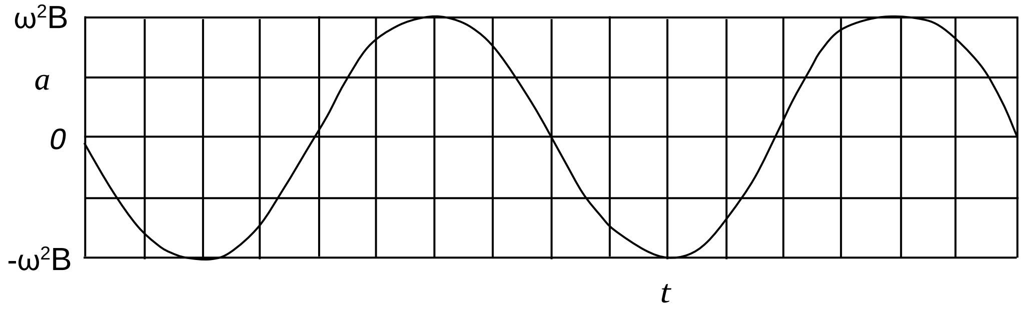
Differentiating, velocity $\quad v = \frac{dx}{dt} = B\omega\cos(\omega t)$

And again acceleration $\quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -B\omega^2\sin(\omega t)$

Here are graphs of x, v and a.







Another fact that appears from the expression for x and a

$$x = B\sin(\omega t) \qquad a = -B\omega^2\sin(\omega t)$$

is that $a = -\omega^2 x$. The acceleration is proportional to the distance from the origin, and is directed towards the origin. This is the classic prescription for simple harmonic motion, usually brought about by the restoring force being proportional to the distance, as with a spring.

Here is an example. Suppose the pendulum of a grandfather clock swings every 2 seconds (the period T is 2 seconds), and the length of the swing from one extreme to the other is 10cm.

Then the amplitude B is 5cm, and the angular frequency ω is $\frac{2\pi}{T}$ which works out as 3.14 (all right, 3.14159265, if you like. But we shall use 3.14 for the moment). Then the motion is described by:

$$x = 5 \sin (3.14t)$$

Differentiating, $v = \frac{dx}{dt} = 5 \times 3.14 \cos (3.14t)$

And again, $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 15.7 \times 3.14 (-\sin (3.14t)$

$$= -49.298 \sin (3.14t)$$

The maximum velocity of the pendulum bob occurs when the cosine is 1, and is 15.7cm/s (0.157m/s, or about one-third of a mile per hour), and the maximum acceleration occurs when the sine is 1 (or -1 if you're fussy), and is 49.298cm/s^2, or about one-twentieth of g.

Exercise 1

The motion of the pistons in the car engine is roughly simple harmonic. A typical stroke (top-to-bottom movement) of a piston is 7cm (that means the **amplitude** of the motion is 3.5cm). A typical maximum engine speed is 6000rpm, or 100 revolutions per second (the frequency).

Write down an equation for x, the distance in cm of the piston from its central position, in terms of the time *t* in seconds. Differentiate to get an expression for the velocity in cm/s.

Differentiate again to get an expression for the acceleration.

Find the maximum velocity of the piston in m/s.

Find the maximum acceleration of the piston. Express the result as number of g, where g is the acceleration due to gravity, and is about 981cm/s^2 or 9.81m/s^2.

Now check your answers.

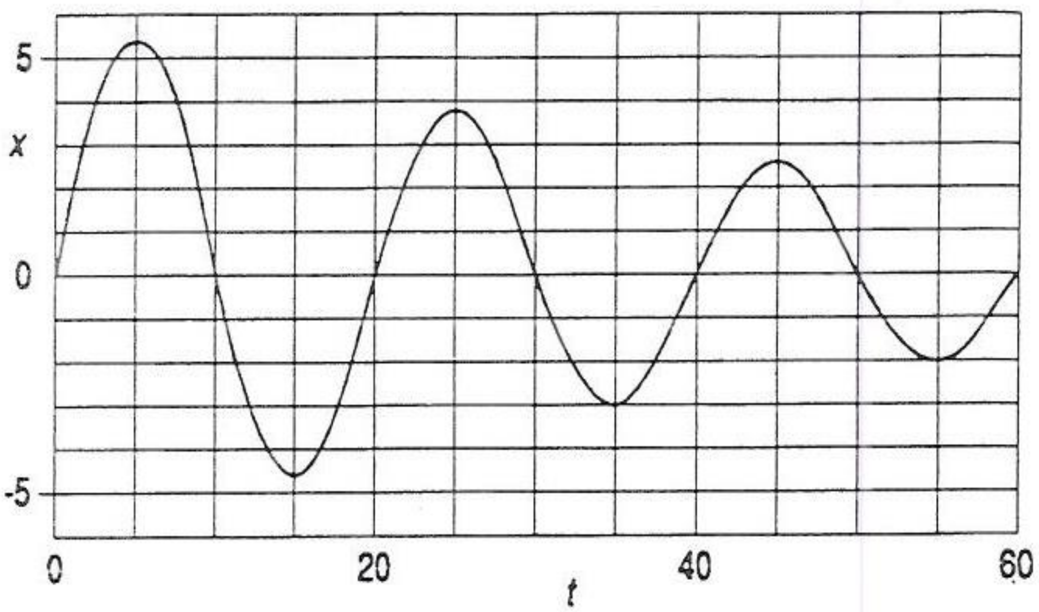
DAMPED HARMONIC MOTION

Unless energy is supplied from an outside source, the normal fate of harmonic motion is to die away. The usual form of dying away is ‘exponential decay’:

$$x = Be^{-kt} \sin(\omega t)$$

Example

$$x = 6e^{-0.02t} \sin(0.314t)$$



The graph shows that the period of the motion is about 20 seconds. The amplitude starts off at 6 and decays away.

The first maximum of the sine wave would usually occur when the angle (0.314*t*) is $\frac{\pi}{2}$. What happens in this case?

Remember, *t* and *x* (and later, u and v) are the variables. All the other letters are constants.

The formula for *x* is a product. The constant B (in this case 6) can be counted as part of one of the functions, or just regarded as a constant multiplier.

$u = e^{-kt}$ $\qquad$ $v = \sin(\omega t)$

$\frac{du}{dt} = -ke^{-kt}$ $\qquad$ $\frac{dv}{dt} = \omega \cos(\omega t)$

(The chain rule was used in both of these differentiations)

$$\frac{dx}{dt} = B\left(v\frac{du}{dt} + u\frac{dv}{dt}\right)$$

$$= B\left[\sin(\omega t)(-ke^{-kt}) + e^{-kt}(\omega\cos(\omega t))\right]$$

So the stationary points are when $\omega \cos(\omega t) = k \sin(\omega t)$

Divide both sides by $\cos(\omega t)$, $\omega = \frac{\sin(\omega t)}{\cos(\omega t)} = k\tan(\omega t)$

So $\tan(\omega t) = \frac{\omega}{k}$

Now put in the figure, $\tan(0.314t) = \frac{0.314}{0.02} = 15.7$

$$0.314t = \tan^{-1}(15.7)$$

As the angle increases from zero, the first angle whose tangent is 15.7 is the one your calculator gives, 1.507 (as usual, we're talking about radians)

$$0.314t = 1.507$$

1.507 is about 0.48π instead of 0.5π

And then $t = \frac{1.507}{0.314} = 4.8$

So the first stationary point occurs after 4.8 seconds.

The graph makes it obvious that the point is a maximum. We can check by finding $\frac{d^{2x}}{dt^2}$

So far the differentiation has been straightforward. If you don't agree, follow it through again.

Don't forget, x and t (and then u and v) are the variables. All the other letters are constant.

$\frac{dx}{dt} = Be^{-kt}(\omega\cos(\omega t) - k\sin(\omega t))$

Put $u = e^{-kt}$ $v = \omega\cos(\omega t) - k\sin(\omega t)$

$\frac{du}{dt} = -ke^{-kt}$ $\frac{dv}{dt} = -\omega^2\sin(\omega t) - k\omega\cos(\omega t)$

$\frac{d^2x}{dt^2} = B\left(v\frac{du}{dt} + u\frac{dv}{dt}\right)$

$= B\,((\omega\cos(\omega t) - k\sin(\omega t))\,(-ke^{-kt})$

$+ e^{-kt}\,(-\omega^2\sin(\omega t) - k\omega\cos(\omega t)))$

So that's the differentiation done.

To do the tidying up, we take out e^{-kt} as a factor, and collect together the sines and cosines:

$$\frac{d^2x}{dt^2} = Be^{-kt}\,(-2\omega k\cos(\omega t) - (\omega^2 - k^2)\sin(\omega t))$$

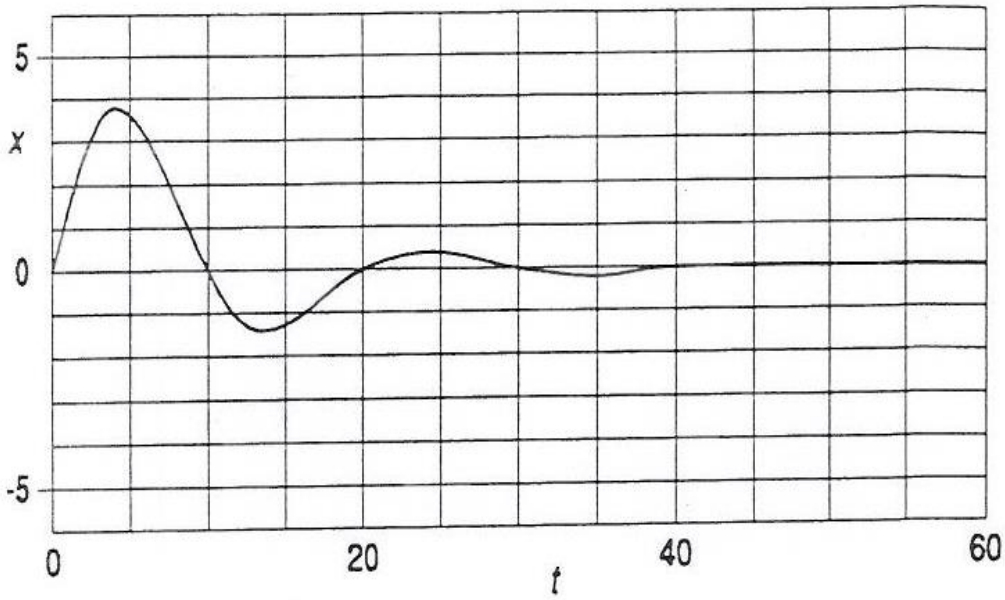
We write '$-(\omega^2 - k^2)$' rather than '$+(k^2 - \omega^2)$' for convenience, because in the current example ω is larger than k. Both ways are correct.

When the angle ωt is less than $\frac{\pi}{2}$ both $\sin(\omega t)$ and $\cos(\omega t)$ are positive, so at the first stationary point $\frac{d^2x}{dt^2}$ is negative. This confirms that it is a maximum.

On the graph at the time of the first maximum at 4.8 seconds can hardly be seen as any different from 5 seconds, which is when the maximum can occur without damping. Here's a case where the difference can clearly be seen:

$x = 6e^{-0.1t}\sin(0.314t)$

Here's the graph.



The graph shows that the motion is more heavily damped. The period of the motion is the same. The first maximum, however, occurs well before the 5 second mark.

Exercise 2

Calculate the time and the size (the x value) of the first maximum.

(You don't have to do any differentiating – just put the new values for B, k and ω into the expressions).

Now check your answers.

These calculations are relevant to many applications where oscillations or 'bouncing' are likely to occur: for example the body of a car after it has hit a bump or the pointer of an instrument which has had a jolt – physical or electrical.

Exercise 3

Here is another damped harmonic motion: $x = Be^{kt} \cos(\omega t)$

Find expressions for $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ (Tidy up the results)

Now check your answers.

SUMMARY

Many motions which we meet from day to day are simple harmonic motions or dampened harmonic options (or nearly so). These are natural vibrations of springy objects such as wires, strings, and pieces of wood or metal, for instance, and vibrations of parts of machines.

The mathematics of harmonic motion are introduced here provide an approach to calculating the movements. This may be done in order to explain observations, or as part of a design process.

ANSWERS

Exercise 1

x = 3.5 sin (200πt)

The rest of the question depends on that. If you got it wrong try to make sure you understand it before continuing. The '200πt' comes from the fact that the 'ω' is equal to 2πf.

$$v = \frac{dx}{dt} = 700\pi \cos(200\pi t)$$

That is 3.5 x 200π x cos (200πt)

$$a = \frac{d^2x}{dt^2} = 14000\pi^2 \sin(200\pi t)$$

So maximum velocity is 700π or 2200cm/s, or 22m/s.

And maximum acceleration a is 14000π^2cm/s^2; about 1381700 cm/s^2, or 1.38 x 10^{6}cm/s^2, or 1.38 x 10^{4}m/s^2.

Dividing the first value by 981, maximum acceleration is 1408.5g.

That is a very high acceleration (humans start passing out at about 6 to 8g), and there will be corresponding high forces (on the con-rod, piston, etc.) Part of the reason for the very high figure is that the acceleration is proportional to the **square** of the rotational speed (the rpm). So if you halve the rpm, you reduce the acceleration to a quarter.

Now return to the text.

Exercise 2

Time of the first maximum t_{max} = 4.02 seconds

Value of the first maximum = 3.82

You can see roughly what the answer should be from the graph. It always helps if you have a rough idea of what the answer should be.

Time t_{max} of first maximum can be found from the equation

$$\tan(\omega t_{max}) = \frac{\omega}{k}$$

So $$\omega t_{max} = \tan^{-1}\left(\frac{\omega}{k}\right)$$

The values given are k = 0.1 ω = 0.314

So $$t_{max} = \frac{1}{0.314}\tan^{-1}\left(\frac{0.314}{0.1}\right)$$

So the time for the first maximum is now near to 4 seconds, instead of the previous value of about 5 seconds.

Now to use this value in the equation for x.

$$x = 6e^{kt}\sin(\omega t)$$

k = 0.1, ω = 0.314, and the value of t to be inserted is 4.02, so

$$\text{maximum } x = 6e^{(-0.1\times4.02)}\sin(0.314 \times 4.02)$$

$$= 3.82$$

Now return to the text.

Exercise 3

$$\frac{dx}{dt} = -Be^{-kt} (k \cos(\omega t) + \omega \sin(\omega t))$$

$$\frac{d^2x}{dt^2} = -Be^{-kt} (2k\omega \sin(\omega t) - (\omega^2 - k^2) \cos(\omega t))$$

If you tend to lose your way a little, go back to the basics and write the working out in full.

$$x = -Be^{-kt} \cos(\omega t)$$

Put $u = e^{-kt}$ $\qquad$ $v = \cos(\omega t)$

$$\frac{du}{dt} = ke^{-kt} \qquad \frac{dv}{dt} = -\omega \sin(\omega t)$$

Just a reminder in case you've forgotten:

to differentiate e^{-kt} put $s = -kt$

$$u = e^{-kt} = e^s \qquad s = -kt$$

$$\frac{du}{ds} = e^s \qquad \frac{ds}{dt} = -k$$

$$\frac{du}{dt} = \frac{du}{ds}\frac{ds}{dt} = e^s(-k) = -ke^{-kt}$$

And the chain rule is also used to differentiate $\cos(\omega t)$

Then $\frac{dx}{dt} = B\left(v\frac{du}{dt} + u\frac{dv}{dt}\right) = B(\cos(\omega t)(-ke^{-kt}) + \ldots$

$$\ldots e^{-kt}(-\omega \sin(\omega t))$$

$$= Be^{-kt}(-k\cos(\omega t) - \omega \sin(\omega t))$$

= the answer given above

Now return to the text.