# MAXIMUM AND MINIMUM 2

## POINT OF INFLECTION

### Example 1

This looks rather simple: $y = x^3$

To find the stationary points: $\frac{dy}{dx} = 3x^2$

So $\frac{dy}{dx}$ is zero when $x = 0$

There is one stationary point, the point (0, 0).

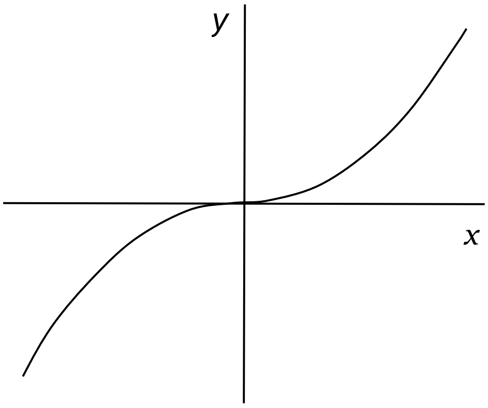
Is it a maximum or a minimum?

When $x$ = 0-, $\frac{dy}{dx}$ is positive.

When $x$ = 0+, $\frac{dy}{dx}$ is positive.

So the curve climbs to the point (0,0) and then climbs away. This is not a maximum or a minimum, but a brief pause in what is otherwise a steadily increasing $y$. So this is a third kind of stationary point. It is called a point of **inflection**.

Here is a sketch of the graph.




### Example 2

Now look at $y = x^4 - 8x^3 + 18x^2 - 7$

To find that stationary points:

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 36x$$

So is zero $\frac{dy}{dx}$ when $4x^3 - 24x^2 + 36x = 0$

(divide by 4) i.e. when $x^3 - 6x^2 + 9x = 0$

i.e. When $x(x^2 - 6x + 9) = 0$

(did you spot the factor $x$? It is one of the points you have to watch out for).

So $\frac{dy}{dx} = 0$ when $x = 0$ or when $x^2 - 6x + 9 = 0$

Now $x^2 - 6x + 9 = (x-3)^2$ which is zero when $x = 3$ (or use the formula). So $\frac{dy}{dx} = 4x(x-3)^2$

Using the formula for $y$, when $x = 0$ $y = -7$, and when $x = 3$ $y = -7$, so the stationary points are (0, -7) and (3, 20).

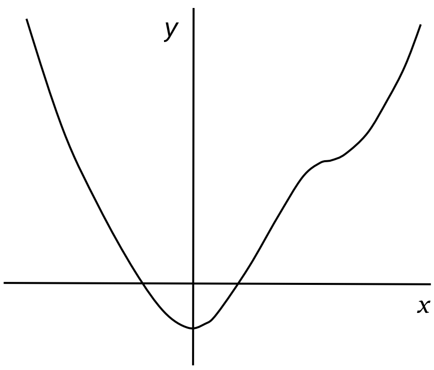
To find what kind of stationary points they are:

$(x-3)^2$ is always positive, so the sign of $\frac{dy}{dx}$ will be the same as the sign of $x$.

So when $x = 0-$ then $\frac{dy}{dx}$ is negative, and when $x = 0+$ $\frac{dy}{dx}$ then is positive, which means that the point (0. -7) is a minimum.

When $x = 3-$ then $\frac{dy}{dx}$ is positive, and when $x = 3+$ then $\frac{dy}{dx}$ is positive, and when $x = 3+$ then $\frac{dy}{dx}$ is positive, which means that the point (3, 20) is a point of inflection.

Here is a sketch of the curve (Additional information has been used to sketch the curve. Curve sketching is covered in a later pack).




Points of inflection will probably not appear very often in your work, but it is important to be aware that they exist.

**Exercise 1**

Find the stationary points of the curve

$$y = x^4 - 2x^3$$

Determine whether each point is a minimum, a maximum or a point of inflection.

Now check your answer.

## TURNING POINTS

Maximum and minimum points are often referred to as **turning points**. A point of inflection is **not** a turning point. All three kinds of points are referred to as stationary points, because a stationary point is any point where $\frac{dy}{dx}$ is zero.

It might help you to remember if you say to yourself: “If the lady’s **not** for **turning** she must be **inflexible**”.

## SUMMARY

A stationary point which is not a minimum or a maximum is called a point of inflection.

A graph continues to increase as it passes through a point of inflection (or, if it is decreasing, it continues to decrease); except that, at the point itself, the rate of change becomes zero.

The gradient of a graph does not change sign as the graph passes through a point of inflection.

**Exercise 2**

The height $h$ in metres of a ball thrown vertically upwards is given by $h = ut - 0.5gt^2$ where $t$ is the time in seconds, $u$ is the throwing speed in m/s and $g$ is the acceleration due to gravity (9.8 $m/s^2$).

What is the throwing speed if the ball just reaches the roof of the house, 7 m height?

(Hint: Find what value of $t$ (in terms of $u$ and $g$) makes $h$ a maximum. Put that value of $t$ into the formula for $h$ to get a formula for the maximum height in terms of $u$ and $g$, then put the figures in and solve it for $u$).

Now check your answer.

**Exercise 3**

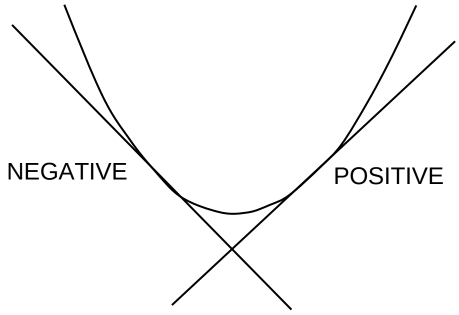
Find the stationary points of the curve $y = x^8 - 8x^5$

Determine whether each point is a maximum, a minimum or a point of inflection.
Now check your answer.

**MAXIMUM AND MINIMUMS**

$\frac{d^2y}{dx^2}$ is a gradient of $\frac{dy}{dx}$. When $\frac{d^2y}{dx^2}$ is positive then $\frac{dy}{dx}$ is increasing; when $\frac{d^2y}{dx^2}$ is negative then $\frac{dy}{dx}$ is decreasing.

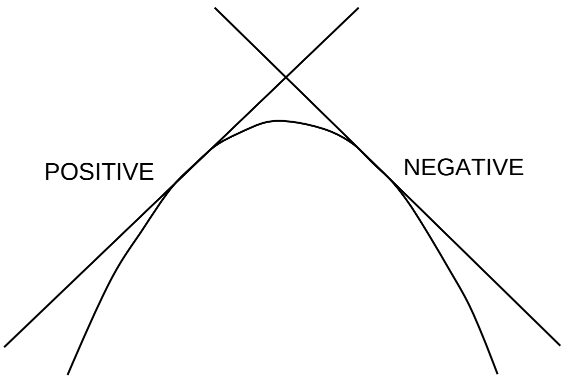
The diagram below shows a minimum on a graph. The tangents demonstrate the gradient $\frac{dy}{dx}$. It is negative before the minimum, zero at the minimum, and positive after the minimum. One way of deciding whether the stationary point is a minimum or a maximum is to check the sign of the gradient before and after the point.




If $\frac{dy}{dx}$ changes from negative to zero to positive then it seems fairly safe to say that it is increasing. So $\frac{d^2y}{dx^2}$ should be positive at a minimum.

By a similar argument, it would seem fairly safe to say that at a maximum $\frac{d^2y}{dx^2}$ should be negative.

So, this is another way of testing a stationary point to see whether it is maximum or a minimum.




If $\frac{d^2y}{dx^2}$ is positive the stationary point is a minimum.

If $\frac{d^2y}{dx^2}$ is negative the stationary point is a maximum.

That makes three ways so far to find out whether a stationary point is a maximum or a minimum.

There are two ‘before and after ‘ways:

test the value of $y$ before and after the point (as well as at the point)

test the sign of $\frac{dy}{dx}$ before and after the point.

And now the third way:

test the sign of $\frac{d^2y}{dx^2}$ at the point.

Take $y = x^2$ as an example. This is (or should be!) a familiar shape, symmetrical with a minimum at the origin.

$$\frac{dy}{dx} = 2x \quad \left(\text{so } \frac{dy}{dx} = 0 \text{ when } x = 0\right)$$

Differentiate again:

$$\frac{d^2y}{dx^2} = 2$$

So when $x = 0$ then $\frac{d^2y}{dx^2}$ is positive, so the point (0, 0) is a minimum.

(In this case $\frac{d^2y}{dx^2}$ is always positive, but we are only concerned with its value when $x = 0$)

**Exercise 4**

In the following cases, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Confirm that the given point is a stationary point (i.e. show that $\frac{dy}{dx} = 0$ at the point), and use $\frac{d^2y}{dx^2}$ to find out whether the point is a minimum or a maximum:

a) $y = 3 + 12x - 2x^2$ Point (3, 21)

b) $y = x^3 + 3x^2 - 9x + 2$ Point (1, -3)

c) $y = x^4 - 12x^2$ Point (0, 0)

d) $y = \frac{18}{x} + 2x$ Point (3, 12)

Having found a stationary point, and found $\frac{d^2y}{dx^2}$, you then have to remember what sign shows that the point is a minimum.

Well, it does not take long to arrive at the answer using a quick sketch. But in the middle of an exam it is easy to make a slip.

Some people find it helps to construct 'silly pictures'. For example mini-pose. A girl in a mini, posing. So a minimum corresponds to $\frac{d^2y}{dx^2}$ being positive.

Or think '$d^2y$ is mi at a high' (dee two wye is my at a high) ('mi' being short for minus).

A more conventional way is simply to remember $y = x^2$. This curve has a single stationary point which is a minimum, and it does not take long to work out that $\frac{d^2y}{dx^2}$ = 2, which is positive.

**The best test?**

There is no final answer to the question, which is the best test? The first criterion is, which is the easiest? In the examples you have just done, the $\frac{d^2y}{dx^2}$ test is the easiest; but if you prefer one of the other ways, use it!

There are occasions when the $\frac{d^2y}{dx^2}$ test does not work. Take the case of $y = x^3$. You will remember that there is a stationary point at the origin, and this point is a point of inflection.

What does $\frac{d^2y}{dx^2}$ tell us about this point?

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

$$\frac{d^2y}{dx^2} = 6x$$

So when $x$ = 0, then $\frac{d^2y}{dx^2}$ = 0

Unfortunately, this is not a final test for an inflection.

The curve $y = x^4$ has a minimum at the origin; and you can easily check that $\frac{d^2y}{dx^2}$ is zero when $x$ = 0 (check it!).

That means that if you find that $\frac{d^2y}{dx^2}$ is zero at a stationary point, you will have to try one of the other methods to find out whether it is a minimum or a maximum or a point of inflection.

For example $y = x^4$

$$\frac{dy}{dx} = 4x^3$$

$$\frac{d^2y}{dx^2} = 12x^2$$

So there is a stationary point when $x$ = 0, because $\frac{dy}{dx}$ = 0 when $x$ = 0.

But $\frac{d^2y}{dx^2}$ is also zero when $x$ = 0.

Examine $\frac{dy}{dx}$. When $x$ = 0-, $\frac{dy}{dx}$is negative; when $x$ = 0+,$\frac{dy}{dx}$ is positive. The point (0, 0) is therefore a minimum.

Now try a few problems. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each case. If $\frac{d^2y}{dx^2}$ is zero, tests the stationary point using the sign $\frac{dy}{dx}$ of before and after.

**Exercise 5**

Find the stationary points of the following curves, and determine whether each point is a minimum, a maximum or a point of inflexion.

a) $y = 2x^6$

b) $y = 12x^2 - 6x$

c) $y = x^3 - 75x$

d) $y = \frac{8}{x^2} + \frac{x^2}{2}$ (there are two stationary points.)

e) $y = x^3 - 12x^2 + 48x - 7$

Now check your answer.

**SUMMARY**

If $\frac{d^2y}{dx^2}$ is negative at a stationary point, the point is a maximum ( $d^2y$ is mi at a high).

If $\frac{d^2y}{dx^2}$ is positive at a stationary point, the point is a minimum.

If $\frac{d^2y}{dx^2}$ is zero at a stationary point, one of the other methods must be used to find out whether the point is a maximum, a minimum or a point of inflection.

## ANSWERS

### Exercise 1

The stationary points are (0,0) and (1.5, -1.6875).

(0, 0) is a point of inflection, and (1.5, -1.6875) is a minimum.

$$y = x^4 - 2x^3 \qquad \frac{dy}{dx} = 4x^3 - 6x^2$$

A factor 2 and a factor $x^2$.

$$\frac{dy}{dx} = 2x^2(2x-3)$$

$\frac{dy}{dx}$ is zero when $x^2 = 0$ or $2x - 3 = 0$

so $x = 0$ or $x = \frac{3}{2}$

When $x = 0$, $y = 0$. When $x = \frac{3}{2}$, $y = \frac{-27}{16}$

Or, if you prefer, when $x = 1.5$, $y = -1.6875$

When $x$ is near zero, $2x - 3$ is negative. $x^2$ is always positive, so $\frac{dy}{dx}$ is negative when $x$ is 0- and $x$ is 0+. So (0,0) is a point of inflection.

When $x$ is near $\frac{3}{2}$, $x^2$ is positive. When $x$ is $\frac{3}{2}$+ is positive.

So the point (1.5, -1.6875) is a minimum.

**Now return to the text.**

**Exercise 2**

The required throwing speed is 11.71 m/s.

Normally it is best to put the figures in, but this case is not straightforward. You are **given** the maximum, and asked to work out the throwing speed $u$. Try leaving the figures out, and dealing with the formula.

$h = ut - 0.5gt^2$ Now to find $\frac{dh}{dt}$

Confusion can arise between constants and variables. Concentrate on $t$ on the right- hand side, and think of all the rest as numbers.

$$\frac{dh}{dt} = u - 0.5g \times (2t) = u - gt$$

$\frac{dh}{dt}$ is zero when $u - gt$ is zero.

After what time is $h$ a maximum?

$u - gt = 0$ gives $t = \frac{u}{g}$

Now find the maximum value of $h$ (we have not proved that it is a **maximum**)
Put $t = \frac{u}{g}$ into the formula for $h$.

$$h_{\max} = u\left(\frac{u}{g}\right) - 0.5\left(\frac{u}{g}\right)^2$$

$$= \frac{u^2}{g} - 0.5\frac{u^2}{g} = 0.5\frac{u^2}{g} = \frac{u^2}{2g}$$

Now we can put in the values $h_{\max} = 7$ and $g = 9.8$.

So $7 = \frac{u^2}{2 \times 9.8} = \frac{u^2}{19.6}$

**Now return to the text.**

**Exercise 3**

Optimum $r$ is 1225. Maximum $F$ is 1.345

$$F = a + \frac{b}{r} + \frac{r}{c} = a + br^{-1} + \frac{r}{c}$$

At this point there is a choice: put the numbers in immediately, or leave them till later? The most reliable approach at this stage is to put them in immediately - the fewer letters there are, the better.

$$F = 1.1 + 150r^{-1} + \frac{r}{10000}$$

$$\frac{dF}{dr} = -150r^{-2} + \frac{1}{10000} = -\frac{150}{r^2} + \frac{1}{10000}$$

So $\frac{dF}{dr} = 0$ when $-\frac{150}{r^2} + \frac{1}{10000} = 0$

$\frac{1}{10000} = \frac{150}{r^2}$. Multiply both sides by $10000r^2$,

$$r^2 = 10000 \times 150$$

$r = 1225$ (to 4 significant figures).

Where $r$ is less than this the negative part of $\frac{dF}{dr}$ gets larger, making it negative; and when $r$ is greater $\frac{dF}{dr}$ becomes positive; so this is a minimum. This value of $r$, 1225, must be the 'optimum source resistance'.

Then $F_{\min} = 1.1 + \frac{150}{1225} + \frac{1225}{10000} = 1.345$

If you noticed that $r = -1225$ is another solution then take an extra mark! However source resistance are normally positive, so ignore that result.

As $x$ increases through the value $\sqrt[3]{5}$, $(x^3 - 5)$changes from negative to positive. So does $\frac{dy}{dx}$. The point (1.710, -43.860) is therefore a minimum.

**Now return to the text.**

**Exercise 4**

a) The point (3, 21) is a maximum. Check your working to confirm that your method was correct.

$$y = 3 + 12x - 2x^2$$

$$\frac{dy}{dx} = 12 - 4x$$

$$\frac{d^2y}{dx^2} = -4$$

When $x = 3$ then $\frac{dy}{dx} = 0$ and $y = 21$. (The fact that $y = 21$ is when $x = 3$ is given in the question, but it is reassuring to confirm it.)

That shows that the point (3, 21) is a stationary point.

When $x = 3$ then $\frac{d^2y}{dx^2}$ is negative. That shows that the point is a maximum.

b) The point (1, -3) is a minimum.

$$y = x^3 + 3x^2 - 9x + 2$$

$$\frac{dy}{dx} = 3x^2 + 6x - 9$$

$$\frac{d^2y}{dx^2} = 6x + 6$$

When $x = 1$ then $\frac{dy}{dx} = 3 + 6 - 9 = 0$

and $y = 1 + 3 - 9 + 2 = -3$

so the point (1 , -3 ) is a stationary point.

When $x = 1$ then $\frac{d^2y}{dx^2} = 6 + 6 = 12$ which is positive.

So the point (1, -3) is a minimum.

c) The point (0, 0) is a maximum.

$$y = x^4 - 12x^2$$

$$\frac{dy}{dx} = 4x^3 - 24x$$

$$\frac{d^2y}{dx^2} = 12x^2 - 24$$

When $x = 0$ then $\frac{dy}{dx} = 0$ and $y = 0$.

So the point (0, 0) is a stationary point.

When $x = 0$ then $\frac{d^2y}{dx^2} = -24$ which is negative.

So the point (0, 0) is a maximum.

d) The point (3, 12) is a minimum.

$$y = \frac{18}{x} + 2x = 18x^{-1} + 2x$$

$$\frac{dy}{dx} = -18x^{-2} + 2$$

$$\frac{d^2y}{dx^2} = 36x^{-3}$$

When $x = 3$ then $\frac{dy}{dx} = -\frac{18}{9} + 2 = 0$

and $y = \frac{18}{3} + 6 = 12$

So the point (3, 12) is a stationary point.

When $x = 3$ then $\frac{d^2y}{dx^2} = \frac{36}{27}$ which is positive.

So the point (3, 12) is a minimum.

**Now return to the text.**

**Exercise 5**

a) The point (0, 0) is a minimum.

We are moving gently towards harder calculus, so if you are still getting things right you can pat yourself on the back. But it is not very hard yet, so if you are having problems you should be able to overcome them by re- reading the text, and if necessary, putting questions to your tutor. Do not forget, try to produce questions rather than just saying 'I cannot do number 3'. It often happens that once you have produced the question, you can answer it yourself.

$$y = 2x^6$$

$$\frac{dy}{dx} = 12x^5$$

$$\frac{d^2y}{dx^2} = 60x^4$$

$\frac{dy}{dx} = 0$ when $x = 0$. When $x = 0$ then $y = 0$

So the point (0, 0) is a stationary point.

But when $x = 0$ then $\frac{d^2y}{dx^2}$ is also zero- no help.

Examine $\frac{dy}{dx}$. When $x = 0-$ then $\frac{dy}{dx}$ is negative.

When $x = 0+$ then $\frac{dy}{dx}$ is positive.

Then point (0, 0) is therefore a minimum.

b) The point $\left(\frac{1}{4}, -\frac{3}{4}\right)$ is a minimum.

$$y = 12x^2 - 6x$$

$$\frac{dy}{dx} = 24x - 6$$

$$\frac{d^2y}{dx^2} = 24$$

$\frac{dy}{dx} = 0$ when $24x - 6 = 0$, so $x = \frac{6}{24} = \frac{1}{4}$ (or 0.25)

When $x = \frac{1}{4}$ then $y = \frac{12}{16} - \frac{6}{4} = \frac{3}{4} - \frac{3}{2} = -\frac{3}{4}$ and $\frac{d^2y}{dx^2}$ is always positive, so the point $\left(\frac{1}{4}, -\frac{3}{4}\right)$ s a minimum.

c) $y = x^3 - 75x$

$$\frac{dy}{dx} = 3x^2 - 75$$

$$\frac{d^2y}{dx^2} = 6x$$

$\frac{dy}{dx} = 0$ when $3x^2 - 75 = 0$, $3x^2 = 75$, $x^2 = 25$

$x = \pm 5$ (don't forget the -5)

When $x = -5$, $y = -125 + 375 = 250$

When $x = 5$, $y = 125-375 = -250$

When $x = -5$, $\frac{d^2y}{dx^2}$ is negative. The point (-5, 250) is a maximum.

When $x = 5$, $\frac{d^2y}{dx^2}$ is positive. The point (5, -250) is a minimum.

d) $y = \frac{8}{x^2} + \frac{x^2}{2} = 8x^{-2} + \frac{1}{2}x^2$

$$\frac{dy}{dx} = -16x^{-3} + x$$

$$\frac{d^2y}{dx^2} = 48x^{-4} + 1$$

$\frac{dy}{dx} = 0$ when $\frac{-16}{x^3} + x = 0$, $x = \frac{16}{x^3}$, $x^4 = 16$, $x = \pm 2$ (don't forget the -2)

When $x = -2$ then $y = \frac{8}{4} + \frac{4}{2} = 4$

and when $x = 2$ then $y = 4$

Since $x^{-4}$ (like $x^4$) is always positive, $\frac{d^2y}{dx^2}$ is always positive.

Therefore, the points (-2,4) and (2, 4) are both minimums.

e) $y = x^3 - 12x^2 + 48x - 7$

$$\frac{dy}{dx} = 3x^2 - 24x + 48$$

$$\frac{d^2y}{dx^2} = 6x - 24$$

$\frac{dy}{dx}$ = 0 when $3x^2 - 24x + 48$ = 0

Divide by 3 $x^2 - 8x + 16$ = 0

$(x-4)^2$ = 0

$x$ = 4 (or do it by formula)

When $x$ = 4 then $y$ = 64- 192 +192 -7 = 57

When $x$ = 4 then $\frac{d^2y}{dx^2}$ = 0

Examine $\frac{dy}{dx}$. $\frac{dy}{dx} = 3(x-4)^2$

Squares are always positive therefore $\frac{dy}{dx}$ is always positive.

So the point (4, 57) is a point of inflection.

**Now return to the text.**