NON-HARMONIC MOTIONS

MOTION IN TREACLE:

Well perhaps not treacle, but some kind of viscous fluid such as oil. Let's first of all take the case of something (like, perhaps, a strawberry floating in cream) which would normally float on the oil or cream or water, but it is given a downward push. Suppose the object is given an initial velocity U downwards. We are going to make x positive upwards; because otherwise graphs of x against time appear to be 'upside down'.

One equation which could describe this is:

$$x = -\frac{(U+V)}{k}(1-e^{-kt}) + Vt$$

In this equation k and V are constants which depend on the viscosity (the 'draginess') of the fluid, the weight of the object, and the buoyancy of the object in the fluid. Then

$$\frac{dx}{dt} = -\frac{(U+V)}{k}(1 - e^{-kt}) + V = (U + V)\, e^{-kt} + V$$

So when t = 0 then the velocity is –(U + V) + V, Which is –U; that is, a velocity of magnitude U downwards. The equation suggests that the downward velocity dies away, and eventually the object has an upward velocity V.

The stationary value of x is when $\frac{dx}{dt} = 0$,

So $-(U + V)\, e^{-kt} + V = 0$

$$V = (U + V)\, e^{-kt} \qquad e^{-kt} = \frac{V}{(U+V)}$$

Now don't forget that $e^{-kt} = \frac{1}{e}$kt so the time of maximum x is given by

$$e^{kt} = \frac{(U+V)}{V} \qquad \text{So } kt = \ln\left[\frac{(U+V)}{V}\right]$$

$$t = \underline{\ln(}\frac{(U+V)}{V})$$

Let's try some figures: put V = 0.5m/s

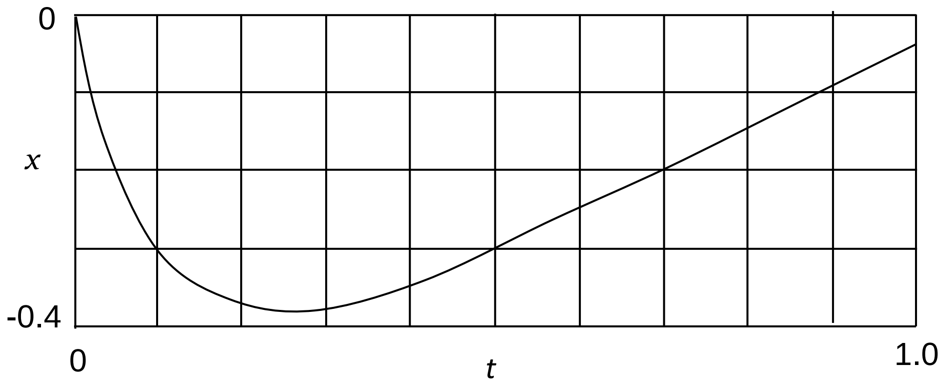
k = 10

initial 'push' velocity U = 5m/s

Then $x = -\frac{(5+0.5)}{10}(1 - e^{-10t}) + 0.5t$

$= -0.55\ (1 - e^{-10t}) + 0.5t$

Here's a graph of that function:



There is no point in continuing the graph beyond the point where x = o, because the object reaches the surface at that point: completely different conditions then apply.

From the calculations, x is a minimum when $t = \dfrac{In\left(\dfrac{U+V}{V}\right)}{k}$

$$= \frac{In\left(\dfrac{5.5}{0.5}\right)}{0.5}$$

$$= 0.24s$$

And the minimum value of x is $-0.55\ (1 - e^{-2.4}) + (0.5)\ (0.24) = -0.38m$

This fits in with the graph.

The practical situation and the graph both make it clear that the stationary value of x is a minimum- a lowest point. But I know that I can't restrain you from differentiating again; so here we go, starting with the equation for $\dfrac{dx}{dt}$

$$\frac{dx}{dt} = -\ (U + V)\ e^{-kt} + V$$

Exercise 1

Write down the expression for $\dfrac{d^2x}{dt^2}$

Now check your answers.

This is always positive given that the constants are positive, confirming that the stationary value of x is a minimum.

Using the given figure, $\frac{d^2x}{dt^2}$ = $55e^{-10t}$

So, for example, the acceleration when t = 0 is $55m/s^2$

Exercise 2

A swimmer jumping from a high board might enter the water at a speed of about 8m/s (U = 8). Plausible values for the other constants are V = 2 and k = 1.2.

Calculate the time to reach maximum depth.

Calculate the maximum depth reached by the swimmer.

Calculate the acceleration as the swimmer enters the water (t = 0) and express it as a multiple of g (9.8m/s).

Now check your answers.

NOT QUITE HARMONIC

This is the kind of motion suited to a sluggish Friday afternoon; or to a well-controlled door; or to a weight on a spring connected to a 'dash-pot' (an item found in old physics laboratories consisting of a pot of oil and a moving plate, intended to damp out oscillations). There is a force trying to restore the subject to its 'normal' position, as in the case of harmonic motion is 'over-damped' so that, after the disturbance, the subject reaches a maximum distance then slowly settles back to its initial position without crossing the zero line.

Suppose the subject is jolted into an initial velocity U (somebody shouts in your ear, or kicks the door, or taps the weight on the spring); here's the equation:

$$x = \frac{U}{(b-a)}(e^{-at} - e^{-bt})$$

Don't forget U, a and b (and e, of course) are all constants; x and t are the variables.

Notice that, when t = 0, both the terms inside the bracket are 1, so the net result is zero.

<u>Exercise 3</u>

Write down the expression for $\frac{dx}{dt}$ given that

$$x = \frac{U}{(b-a)}(e^{-at} - e^{-bt})$$

(Don't forget, $\frac{U}{(b-a)}$ is just a constant multiplier).

Now check your answers.

When you put t = 0 into this expression, you should find that it works out to U, the initial velocity. Try and see.

To find when x is stationary, put $\frac{dx}{dt}$ = 0.

Look at the expression inside the bracket:

$$-ae^{-at} + be^{-bt} = 0$$

$$be^{-bt} = ae^{-at}$$

Multiply both sides by e^{bt}. B = $ae^{(b-a)t}$

($e^{-bt}\, x e^{bt} = e^0 = 1$; $e^{-at}\, x\, e^{bt} = e^{-at + bt} = e^{(b-a)t}$)

Divide both sides by a (and change sides):

$$e^{(b-a)t} = \frac{b}{a}$$

Take logs, $(b – a)t = \ln\left(\frac{b}{a}\right)$

$$t = \frac{In(\frac{b}{a})}{(b-a)} = \frac{In(b) - In(a)}{(b-a)}$$

Just to make sure that this is a maximum and also, for the sheer pleasure of it, let's differentiate again:

Exercise 4

Given that $\frac{dx}{dt} = \frac{U}{(b-a)}(-ae^{-at} + be^{-bt})$

Find the expression for $\frac{d^2x}{dt^2}$

Now check your answer.

Well, it would be possible now to insert the expression for t which gives stationary x, and show that $\frac{d^2x}{dt^2}$ is negative; but the working has been a bit heavy (though not excessively), so we'll leave that for the moment.

Let's try with some numbers. Take the case of a self-closing door. Suppose an uncouth person, careless of whether any unsuspecting fellow-human is approaching from the other side, gives the door a hefty boot. In this case x will have to be the **angle** of the doors in radians, starting at zero.

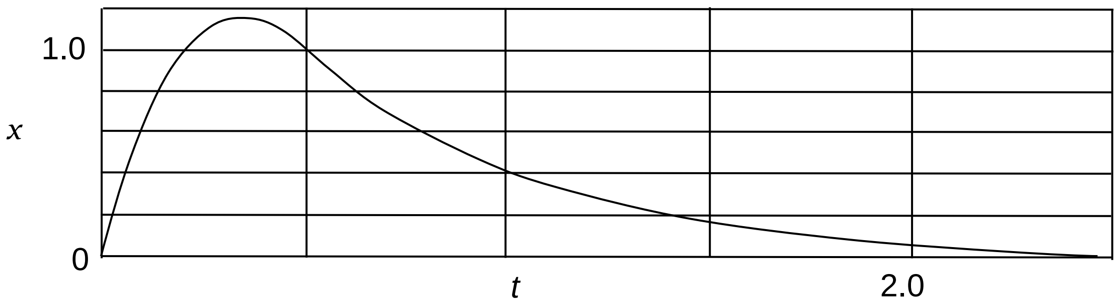
Initial angular velocity of door U = 10 rad/s

a = 2.29 b = 4.31 so (b-a) = 2.02

Then $x = \frac{10}{2.02}(e^{-2.29t} - e^{-4.31t})$

$= 4.95\,(e^{-2.29t} - e^{-4.31t})$

Here's a graph of the function:



This shows the subject adjusting gently back towards its original position, rather than, as in the previous graph, simply adopting a constant velocity.

Exercise 5

Calculate the time when the angle x of the door has a stationary value.

Confirm that this is a maximum by showing that $\frac{d^2x}{dt^2}$ is a negative at that time.

Calculate the maximum angle of the door. (use whatever results you need from the above calculations).

Now check your answers.

NOT QUITE BUT ALMOST HARMONIC

When the damping is just enough to prevent the subject oscillating (crossing the zero axis), it is known as critical damping. Here's the equation of a critically damped motion:

$x = Ute^{-at}$

As usual, U and a are constants. When differentiating this, you will recognise it is as a product.

Exercise 6

Given that x = Ute^{-at} find an expression

for $\frac{dx}{dt}$ and show that constant U is in fact the initial velocity (the velocity when t = 0).

If U = 10 and a = 3.2, calculate the time when x has a stationary value.

Calculate the stationary value of x.

Find an expression for $\frac{d^2x}{dt^2}$ and use it to show that the stationary value of x is in fact a maximum.
(Although the equation looks different, this motion is in fact very similar to the motion of the door in the previous example).

Now check your answers.

SUMMARY

Some motions of bodies in a resisting medium can be described quite well using exponentials.

The motions may involve uniform forces such as gravity. There may also be forces towards a central point such as the type of force that produces harmonic motions; but the resistive medium damps the motion so that no oscillation occurs.

ANSWERS

Exercise 1

$$\frac{d^2x}{dt^2} = k\,(U + V)\,e^{-kt}$$

That should be fairly straightforward operation by now, after all the hard one's you have done.

$$\frac{dx}{dt} = -(U + V)\,e^{-kt} + V$$

The V itself is a constant, so it disappears when you differentiate. -(U + V) is a constant multiplier, so it remains to differentiate e^{-kt}. And that gives $-ke^{-kt}$.

So $\frac{d^2x}{dt^2} = -(U + V)(-ke^{-kt}) = k\,(U + V)\,e^{-kt}$

The constant k, U and V are all positive, and e^{-kt} is always positive, so $\frac{d^2x}{dt^2}$ is always

(though it gets very close to zero when t gets large).

Now return to the text.

Exercise 2

Time to minimum is 1.341s.

Minimum x is -3984m (lowest point below surface).

Acceleration on entry to water is 1.224g.

Here are the workings:

The time of minimum is given by: $$t = \frac{In\left(\frac{U+V}{V}\right)}{k}$$

$$= \frac{In\left(\frac{10}{2}\right)}{1.2}$$

$$= 1.341\text{s}$$

And then $x = -\frac{(U+V)}{k}(1 - e^{-kt}) + Vt$

$= -\frac{10}{1.2}(1 - e^{-(1.2 \times 1.341)}) + 2 \times 1.341$

$= -3.984$m

You worked out the formula for the acceleration $\frac{d^2x}{dt^2}$

$\frac{d^2x}{dt^2} = k(U + V)e^{-kt}$

When t = 0 then $\frac{d^2x}{dt^2} = k(U + V) = 1.2(8 + 2) = 12\text{m/s}^2$

This is the equivalent to $\frac{12}{9.8}$g = 1.224g.

This upward acceleration is brought about by the buoyancy of the swimmer in the water and the resistance of the water.

Now return to the text.

Exercise 3

$$\frac{dx}{dt} = \frac{U}{(b-a)}(-ae^{-at} + be^{-bt})$$

It's a matter of concentrating on terms inside the brackets. Differentiating e^{-at} (with respect to t) gives –ae^{-at}, and so on.

Now return to the text.

Exercise 4

$$\frac{d^2x}{dt^2} = \frac{U}{(b-a)}(a^2e^{-at} - b^2e^{-bt})$$

If you didn't get this right try again concentrating on the terms inside the brackets:

$$\frac{dx}{dt} = \frac{U}{(b-a)}(ae^{-at} - be^{-bt})$$

As before differentiating e^{-at} gives $-ae^{-at}$ and so on; so it shouldn't be too hard.

Now return to the text.

Exercise 5

The time is t = 0.313s

Ignoring the constant multiplier, the terms inside the bracket come to 2.56 – 4.82, which is negative.

The value of x at this time is 1.133 (about 65 degrees).

Notice that the time and the maximum x agree with the graph.

If you didn't get them, try again. Use the results given in the text. For example, the time of the stationary value is given by

$$t = \frac{In(\frac{b}{a})}{(b-a)}$$

$$= \frac{In(\frac{4.31}{2.29})}{2.02} = 0.313s$$

Here's the expression for $\frac{d^2y}{dx^2}$ with the numbers inserted:

$$\frac{d^2x}{dx^2} = 4.95\ (2.29^2\ e^{-2.29t} - 4.31^2\ e^{-4.31t})$$

The question is, is this negative or positive when t = 0.313? Only the part inside the bracket need be worked out; and it comes to 2.56 – 4.82, which is negative.

To find the maximum value of c, use the expression for x:

$$x = 4.95\ (e^{-2.29t} - e^{-4.31t})$$

and put t = 0.313. Then $x = 4.95\ (0.4883 - 0.2595) = 1.133$

(the two terms inside the bracket are shown as calculated separately; but the bracket and, in fact, the whole calculation should be done in the calculator, without having to write down any intermediate values).

Now return to the text.

Exercise 6

$$\frac{dx}{dt} = Ue^{-at}(1 - at)$$

When t = 0 then $\frac{dx}{dt} = Ue^{0} \times 1 = U$

$$\frac{dx}{dt} = 0 \text{ when } t = \frac{1}{3.2} = 0.3125\text{s}$$

At that time x = 1.150

$$\frac{d^2x}{dt^2} = aUe^{-at}(at-2)$$

Your answer may not be in exactly the same form as that. Check the working to see some other possibilities. Or you may have decided to put the figures in before differentiating. That's perfectly OK, here it is;

$$\frac{d^2y}{dx^2} = 32e^{-32t}(3.2t - 2)$$

If you got any of these wrong check through your working.

Here's the working: $x = Ute^{-at}$

Use the product rule:

Put s = t w = e^{-at}

(avoid using u and v, because they both could cause confusion here)

$$\frac{ds}{dt} = 1 \qquad \frac{dw}{dt} = -ae^{-at}$$

$$\frac{dx}{dt} = U\left(w\frac{ds}{dt} + s\frac{dw}{dt}\right)$$

$$= U(e^{-at}(1) + t(-ae^{-at}))$$

$$= U(e^{-at} - ate^{-at})$$

$$= Ue^{-at}(1 - at)$$

When you put t = o this works out to be U.

$\frac{dx}{dt}$ = 0 when $Ue^{-at}(1 - at) = 0$

Which occurs only when 1 –at = 0, at = 1, t = $\frac{1}{a}$

Given that a = 3.2, then t = $\frac{1}{3.2}$ = 0.3125

$x = Ute^{at} = 10te^{-3.2t}$

When t = 0.3125 then we know that 3.2t = 1, so

$x = 10 \times 0.3125 \times e^{-1} = 1.150$

To find $\frac{d^2x}{dt^2}$ use the product rule again on $\frac{dx}{dt}$:

$\frac{dx}{dt} = Ue^{at}(1 - at)$

Put s = e^{-at} w = 1 – at

$\frac{ds}{dt} = -ae^{-at}$ $\frac{dw}{dt} = -a$

$\frac{d^2x}{dt^2} = U\left(w\frac{ds}{dt} + s\frac{dw}{dt}\right)$

$= U\left((1-at)(-ae^{-at}) + e^{-at}(-a)\right)$

$= U(-ae^{-at} + a^2te^{-at} - ae^{-at})$

$= U(a^2te^{-at} - 2ae^{at})$

$= aUe^{-at}(at-2)$

Now at = 1 gives the stationary value of x, and that makes this expression negative (a, U and e^{at} are all positive); so x is a maximum.

Now return to the text.