EQUATION OF A CIRCLE (COMPLETING THE SQUARE)

Unfortunately, you may not always be given the equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$. They can be written by using a method called completing the square.

The process for this is quite mechanical and is as follows:

For example, complete the square of $x^2 + 2x$

$x^2 + 2x$

$\downarrow \quad \downarrow$ $\frac{1}{2}$ the coefficient of x

$(x + 1)^2$

Check this $(x + 1)^2 = x^2 + 2x + 1$ so we have + 1 too much so this needs to be taken off

So $x^2 + 2x = (x + 1)^2 - 1$

Example 1

$x^2 + 4x + 1$

$\downarrow \quad \downarrow$

$(x + 2)^2$

$(\frac{1}{2}$ the coefficient of $x)$

Expand the bracket $(x + 2)^2 = x^2 + 4x + 4$

We want $x^2 + 4x + 1$ so we need to subtract 3

So $x^2 + 4x + 1 = (x + 2)^2 - 3$

Note: The coefficient of the squared term must equal 1.

Example 2

$2y^2 + 8y + 4$

This needs to be written $2(y^2 + 4y + 2)$

$\downarrow \quad \downarrow$ $\frac{1}{2}$ coefficient of y

So $(y + 2)^2$

Although we have left out the multiplying factor 2 we must remember to replace it later.

$(y + 2)^2 = y^2 + 4y + 4$ we require $y^2 + 4y + 2$

So need to subtract 2

$y^2 + 4y + 2 = (y + 2)^2 - 2$

$\therefore \quad 2(y^2 + 4y + 2) = 2(y + 2)^2 - 2$

Exercise 1

Complete the square for the following:

a) $x^2 + 3x + 1$

b) $2x^2 + 3x + 5$

Now check your answer.

Now we'll look at how completing the square can be used to get equations into our standard form.

Example 3

Find the centre and radius of the equation $x^2 + y^2 - 2x - 4y = 0$

We want to get this equation into the form

$(x - a)^2 + (y - b)^2 = r^2$

First put the x's and y's together

$x^2 - 2x + y^2 - 4y = 0$

Now complete the square for the x's and then for the y's

Let's start with: $x^2 - 2x$

From this we get: $(x - 1)^2$

Expanding we get: $(x - 1)^2 = x^2 - 2x + 1$ so I need to subtract 1

Thus: $x^2 - 2x = (x - 1)^2 - 1$

Now $y^2 - 4y$

From this we get: $(y - 2)^2$

Expanding we get: $(y - 2)^2 = y^2 - 4y + 4$ so I need to subtract 4

Thus: $y^2 - 4y = (y - 2)^2 - 4$

So now I can write:

$x^2 - 2x + y^2 - 4y = 0$

as $(x - 1)^2 - 1 + (y - 2)^2 - 4 = 0$

rearranging $(x - 1)^2 + (y - 2)^2 = 5$

This is our standard form giving centre (1, 2) radius $\sqrt{5}$

Example 4

Find the centre and radius of the circle whose equation is

$2x^2 + 2y^2 - 8x + 6y + 5 = 0$

Firstly I must divide by 2 to make the coefficients of x^2 and y^2 unity before I complete the square

$x^2 + y^2 - 4x + 3y + \frac{5}{2} = 0$

Collect together x's and y's:

$x^2 - 4x + y^2 + 3x + \frac{5}{2} = 0$

Complete the square.

First: $x^2 - 4x$

$(x - 2)^2 = x^2 - 4x + 4$ so I must subtract 4

$x^2 - 4x = (x - 2)^2 - 4$

Now $y^2 + 3x$

$(y + \frac{3}{2})^2 = y^2 + 3x + \frac{9}{4}$ so I must subtract $\frac{9}{4}$

So $y^2 + 3x = (y + \frac{3}{2})^2 - \frac{9}{4}$

Therefore I can write $x^2 + y^2 - 4x + 3y + \frac{5}{2} = 0$

as: $(x - 2)^2 - 4 + (y + \frac{3}{2})^2 - \frac{9}{4} + \frac{5}{2} = 0$

or: $(x - 2)^2 + (y + \frac{3}{2})^2 - \frac{15}{4} = 0$

$(x - 2)^2 + (y + \frac{3}{2})^2 = \frac{15}{4}$

So the circle has centre $(2, -\frac{3}{2})$ radius $(\frac{15}{4})^{1/2}$

Exercise 2

Find the centre and radius of the circle whose equation is

$x^2 + y^2 - 8x - 4y = 0$

Now check your answers.

Exercise 3

Find the centre and radius of the circle whose equation is

$x^2 + y^2 + 3x + 2y + 1 = 0$

Now check your answers.

Exercise 4

Find the centre and radius of the circle whose equation is

$2x^2 + 2y^2 + 6x + 4y + 4 = 0$

Now check your answers.

Exercise 5

Find the centre and radius of the circle whose equation is

$3x^2 + 3y^2 + 2x = 0$

Now check your answers.

ANSWERS

Exercise 1

a) $x^2 + 3x + 1$ The coefficient of x^2 is unity so we can complete the square immediately.

$$(x + \frac{3}{2})^2$$

$(x + \frac{3}{2})^2 = x^2 + 3x + \frac{9}{4}$ I want $x^2 + 3x + 1$ so I need to subtract $\frac{5}{4}$

$\therefore\ x^2 + 3x + 1 = (x + \frac{3}{2})^2 - \frac{5}{4}$

b) $2x^2 + 3x + 5$ I need to factorise by 2 to get the coefficient of x^2 unity.

$2\ (x^2 + \frac{3}{2}x + \frac{5}{2})$

Just consider $(x^2 + \frac{3}{2}x + \frac{5}{2})$

$$(x + \frac{3}{4})^2$$

$(x + \frac{3}{4})^2 = x^2 + \frac{3}{2}x + \frac{9}{16}$ but I want $x^2 + \frac{3}{2x} + \frac{5}{2}$

So need to add $\frac{31}{16}$

So $2\ (x^2 + \frac{3}{2}x + \frac{5}{2}) = 2\ \{(x + \frac{3}{4})^2 + \frac{31}{16}\}$

Now return to the text.

Exercise 2

The correct answer is centre (4, 2) radius $\sqrt{20}$ Look at solution below.

$x^2 + y^2 – 8x – 4y = 0$

$x^2 – 8x + y^2 – 4y = 0$ collecting x's and y's

$x^2 – 8x$	$y^2 – 4y$
$(x – 4)^2 = x^2 – 8x + 16$	$(y – 2)^2 = y^2 – 4y + 4$
	Completing the square
$\therefore\ x^2 – 8x = (x – 4)^2 – 16$	$\therefore y^2 – 4y = (y – 2)^2 – 4$

So $x^2 + y^2 – 8x – 4y = 0$

Can be written $(x – 4)^2 – 16 + (y – 2)^2 – 4 = 0$

Or $(x – 4)^2 + (y – 2)^2 – 20 = 0$

$(x -4)^2 + (y – 2)^2 = 0$

centre (4, 2) radius $\sqrt{20}$

Now return to the text.

Exercise 3

The correct answer is centre $(\frac{-3}{2}, -1)$ radius $\frac{3}{2}$ Look at the solution below.

$x^2 + y^2 + 3x + 2y + 1 = 0$

$x^2 + 3x + y^2 + 2y + 1 = 0$

$(x + \frac{3}{2})^2 - \frac{9}{4} + (y + 1)^2 – 1 + 1 = 0$ completing the square

So $(x + \frac{3}{2})^2 + (y + 1)^2 - \frac{9}{4} - 1 + 1 = 0$

$(x + \frac{3}{2})^2 + (y + 1)^2 - \frac{9}{4} = 0$

or $(x + \frac{3}{2})^2 + (y + 1)^2 = \frac{9}{4}$

so centre $(\frac{-3}{2}, -1)$ radius $\frac{3}{2}$

Now return to the text.

Exercise 4

First divide through by 2. If you didn't do this do the activity again before reading on.

$$x^2 + y^2 + 3x + 2y + 2 = 0$$

$x^2 + 3x + y^2 + 2y + 2 = 0$ collecting x's and y's

$(x + \frac{3}{2})^2 - \frac{9}{4} + (y + 1)^2 - 1 + 2 = 0$ completing the square

$(x + \frac{3}{2})^2 + (y + 1)^2 - 1 - \frac{9}{4} + 2 = 0$

$(x + \frac{3}{2})^2 + (y + 1)^2 - \frac{5}{4} = 0$

$(x + \frac{3}{2})^2 + (y + 1)^2 = \frac{5}{4}$

Centre $(\frac{-3}{2}, -1)$ radius $(\frac{5}{4})^{1/2}$

Now return to the text.

Exercise 5

This one may seem harder from the previous ones. In fact, because there is no y term life is easier!

First divide through by 3

$$x^2 + y^2 + \frac{2x}{3} = 0$$

Collect the x's together

$$x^2 + \frac{2x}{3} + y^2 = 0$$

Complete this square. (There is no need for y)

$(x + \frac{1}{3})^2 - \frac{1}{9} + y^2 = 0$

$(x + \frac{1}{3})^2 + y^2 = \frac{1}{9}$

So centre $(-\frac{1}{3}, 0)$ radius $\frac{1}{3}$